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THE
PRINCIPLES AND APPLICATION
OF
IMAGINARY QUANTITIES,
BOOK II;

DERIVED FROM
A PARTICULAR CASE
OF
Functional Projections:
BEING THE SECOND OF A SERIES OF
ORIGINAL TRACTS
ON VARIOUS PARTS OF THE MATHEMATICS.

By BENJAMIN GOMPERTZ, Esq.

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To EPHRAIM GOMPERTZ, *Esq.*

My dear Brother,

I FEEL pleasure in inscribing this Tract to you, arising from various sources, but from which I shall only particularize—the gratifying recollection of the time in which, free from occupations inimical to the sciences, we endeavoured, and believed ourselves successful in the attempt, to reap reciprocal benefit from the comparison of our separate ideas,—your wish that these Essays should be published,—and my own inclination to convince you that I esteem the results of your judgment.

Your's, &c.

Benjamin Gompertz.

ADDITIONAL ERRATA TO BOOK I.

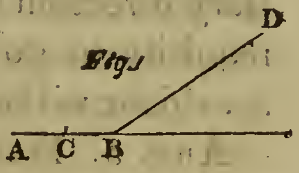
Page 16, line 8, for 1 read 3. Page 17, line 4, for $-\varrho\sqrt{\varrho-\frac{3}{4}}-\frac{3}{2}\varrho$ read $\varrho\sqrt{\varrho-\frac{3}{4}}+\frac{3}{2}\varrho$. Page 18, line 8, for $\sqrt{-\frac{3}{4}}$ read $\sqrt{\varrho-\frac{3}{4}}$. Page 22, line 5, for hyp. log. of $\frac{\sqrt{-1}}{\sqrt{-1}}$ read $\frac{\text{hyp. log. of } \sqrt{-1}}{\sqrt{-1}}$. Page 26, line 15, for 76 read 72.

INTRODUCTION.

Art. 1. An algebraic expression, in which any letter, x for instance, is concerned, is said to be a function of x; thus, $a-x$, $a+x$, $ax+bx^2$, $\sqrt{2ax+x^2}$, &c. are all said to be functions of x. Any expression made up of two or more letters may, in like manner, be said to be a function of any or all of those letters; thus, $\sqrt{2ax+x^2}$ may be said to be a function of x, or a function of a and x.

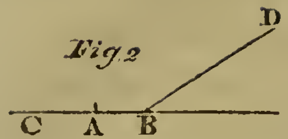
Art. 2. In the application of algebra to geometry, suppose the points A, C, B, to be in a right line, and suppose $AC=x$, $BC=a$, and conceive the right line BD [*Fig.* 1], by some property proposed in the inquiry, to be expressible in terms of AB, that is, in terms of $a+x$; then may BD be said to be a function of $a+x$, or it may be said to be a function of x, or a function of a; but unless a and x are functions of each other, calling such an expression only a function of one of these quantities would only be proper, under the idea of our meaning simply to draw the attention to the manner in which the said quantity enters in the expression. The same is to be noticed with regard to Art. 1.

Fig. 1



Art. 3. If there are two expressions or functions, one in which y is concerned without x, and the other in which x is concerned without y; and if the former, by writing x in the room of y, becomes identical with the latter, then is the former said to be the same function of y as the latter is of x, or the former may be said to be a function of the same *character* with regard to y, that the latter is with regard to x; thus, for instance, $\sqrt{2ay+y^2}$ is the same function of y that $\sqrt{2ax+x^2}$ is of x; and we would also say, that $\sqrt{2ay+y^2}$ is the same function of a and y as $\sqrt{2bx+x^2}$ is of b and x: and here we must be careful not to transpose; for instance, not to say $\sqrt{2ay+y^2}$ is the same function of a and y, that $\sqrt{2bx+x^2}$ is of x and b.

Art. 4. Returning to the example of the application of algebra to geometry (Art. 2), in which BD is supposed expressible by BC, we



observe that, if the point C [*Fig.* 2], instead of being taken at the same side of A that B lies, as in Art. 2, we take it on the contrary side; AB, instead of being $=a+x$, as in that Art., will be $=a-x$; and other things remaining as before, BD will now be the same function of $a-x$ as it was in Art. 2 of $a+x$; or, if we are only anxious to particularise the function with regard to the manner in which x enters, we say BD, in this Article, is the same function of $-x$, that BD, in Art. 2, is of x; and consequently we see, that if x, in the solution of a geometrical problem, had been put for a line tending in a certain direction from a point, and x comes out negative, it is sufficient, to give significance to the solution, to consider the line to tend from the said point in the contrary direction to that assumed in the solution; and that it is not at all requisite to go through a new solution under that change, as the effect is evident without.

Wallis

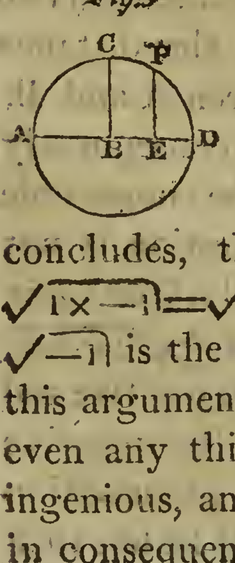
Art. 5. From the explanation of the use of negative roots in the application of algebra to geometry, curiosity leads us to the inquiry, whether or not imaginary roots can have also a meaning in geometrical researches; and our Dr. Wallace, to whom analysis is much indebted, in his Algebra has many attempts to explain the meaning the imaginary roots may have in the application of algebra to geometry. Among other places, he says, p. 268, "So that, whereas, in case of "negative roots, we are to say that the point B cannot be found, so as is "supposed, in AC forward, but backward from A it may in the same "line; we must here say, in the case of a negative square, the point B "cannot be found, so as was supposed, in the line AC, but above that line "it may, in the same plane. This I have the more largely insisted on, "because the notion, I think, is new, &c." There is, in my opinion, much ingenuity in this observation, though I do not think it correct. If a problem requires the position of a point, it is not at all unlikely that, in attempting a solution of the problem, we might have presumed the point to be situate in a line where it cannot be, notwithstanding the question might in itself be possible; and in that case, provided the resulting equation rests upon that presumption, the said equation ought to indicate the impossibility of the presumption; and from this we at least learn that we are not to consider a problem to be impossible from the bare knowledge of all the roots of the resulting equation being impossible, till we have shewn that we have not limited the question in our attempts by our presumption; and, whatever we may determine on Dr. Wallace's ideas, it must be granted that it would be a brilliant discovery, could the roots of a resulting equation, under any limitation of presumption in the investigation, be shewn to lead, at once, either to the solution of the problem or to a demonstration of the impossibility of it. If a point sought is to be in a certain right line, though we may have presumed that the required point (see Art. 4) is on the wrong side of a certain given point, which is the origin of the magnitude represented by the unknown quantity, a negative value of the unknown quantity may correct the hypothesis; still it may happen, notwithstanding all the roots of the equation under consideration are impossible, that the question may be possible, and that the point may be really found in the said right line; and from this we might expect, in case the impossible roots of an equation could be rendered useful in determining the position of a point required by the question, as the learned Dr. Wallace suspected, that there is no reason for expecting, as he did, that such roots particularly direct the point to be taken out of the line in which the solution presumed it. That my meaning may be perfectly comprehended, it may be expedient to consider what now follows.

Art. 6. Let it be required to find a point C in the same line with two given points AB, so that the rectangle AC.BC may $=b^2$, the distance AB being $=a$. Now, if the point C be presumed between A and B, and AC be put $=x$, we shall have $BC=a-x$, and consequently $ax-x^2=b^2$; and therefore $x=\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2-b^2}$, which will be impossible when b^2 is greater than $\frac{1}{4}a^2$; notwithstanding which the question is not impossible, for if the points be taken in the order A, B, C, and AC be again put $=x$, we shall have the equation $ax+x^2=b^2$, and $x=-\frac{1}{4}a\pm\sqrt{\frac{1}{4}a^2+b^2}$, and the two roots which this gives are always possible, and the positive and negative roots both answer the conditions of the question, the negative root giving the order of the points C, A, B, contrary to the presumption in the solution: and this remark is by no means new. If b^2 be less than

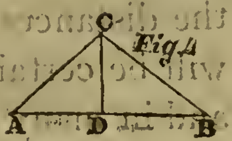
$\frac{1}{4}a^2$, then we have four real answers to the question; but, in order to obtain these by the common rules of algebra, we are obliged to solve the question by two different methods. Whilst some questions require two or more different solutions to arrive at all the answers, there are others of which there are roots to the resulting equation of which the question will not admit; as, if the question required a certain addition, and some of the roots of the equation were, in consequence of their coming out negative, to convert the operation into a subtraction; or, if the question in any way limited the magnitude of a line in a manner not determinable from the equation, &c.

ART. 7. An attempt to explain the use of imaginary quantities in geometry, very similar to that of Dr. Wallace, is one of a very ingenious gentleman, Monsieur l'Abbé Buée, in the Philosophical Transactions for the Year 1806, a work which I read when it was first published, and, being then unacquainted with what Dr. Wallace had written on the subject, I conceived the paper well worth serious attention; and though I have not been able to coincide with that very ingenious gentleman's ideas, I am anxious to say that I owe many of my observations in the present book to my endeavours to follow the idea contained in that work. The Abbé considers, as did Dr. Wallace before him, $\sqrt{-1}$ a mean proportion between $+1$ and -1; and conceiving that if a line in any one direction be represented 1, that an equal line in the contrary direction will be represented by -1. Moreover, that in a circle ACFD [*Fig.* 3], if AD be a diameter, and FE a right line cutting the circle in F and the diameter AD in E, perpendicular to AD, we have $FE = \sqrt{AE.ED}$, it follows that if BC be perpendicular to AD, and pass through the centre B, we shall have $BC = \sqrt{AB \times BD}$. The Abbé concludes, that, if $BD = 1$, AB being $= -1$, BC will be equal to $\sqrt{1 \times -1} = \sqrt{-1}$, BC being 1 in magnitude, and from thence that $\sqrt{-1}$ is the sign of perpendicularity. I am willing to confess that this argument by no means appears to me to carry evidence with it, or even any thing in the shape of evidence; still the object seems to be ingenious, and the use made of it worthy of consideration; and I shall, in consequence, now treat on the application made of this by our author,

Fig. 3



to explain the meaning of the imaginary root of the equation in Art. 6 taking place when b^2 is greater than $\frac{1}{4}a^2$; and as x is found $=\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2-b^2}$, we may write it $x=\frac{1}{2}a\pm\sqrt{b^2-\frac{1}{4}a^2}\times\sqrt{-1}$; and we are, in consequence of the idea of the author [*Fig.* 4], to take $AD=\frac{1}{2}a$ in the right line AB, and then to erect DC perpendicular to it and equal to $\sqrt{b^2-\frac{1}{4}a^2}$, to join AC, BC, and to have the rectangle $AC \cdot BC=b^2$, which is true; and, to use the expression of Dr. Wallace, we have a solution to the question proposed in Art. 6, by allowing a mitigation, which mitigation is only leaving out the limitation that C *must* be in the same right line with AB. Here I remark, that one and the same equation may result from various questions; that it would be easy to propose questions to which any proposed equation may be the resulting equation, and such that the imaginary roots would give solutions if explained in the above manner; but other questions might be proposed to which the same equation would result where such an explication would bring out a false conclusion; and this I think a sufficient plea to reject it. I might likewise add, that the question in Art. 6, with the mitigation here proposed, would be unlimited, and that therefore a solution which limits it cannot be what is sought after. The learned author's idea seems to me, though not declared, to be that a line in one direction may be expressed by a certain function of an equal line in another direction; that if the first line be represented by a, that the line in the opposite direction would be represented by $-a$; and if, for instance, we represent BD [*Fig.* 3] by a, and BC by such a function of a as is denoted by ma, m being a given quantity, that, in consequence of similar positions, AB would be the same function of BC that BC is of BD, and consequently $AB=m.BC=m^2.BD$; but AB is said to be equal to $-a$, and $BD=a$, and therefore $m^2=-1$, or $m=\sqrt{-1}$; whence $BC=\sqrt{-1}.a$. The Reader must not forget that we have begged the question in supposing that the required function which BC is of BD was necessarily expressed by ma, and that such a solution is not to be trusted to. Notwithstanding this, I have, in consequence of this theory, been led to consider the application of $\sqrt{-1}$ in geometry more minutely, and have fallen on the idea to consider it as resulting from



a particular case of *functional projection;* and, to prepare the Reader's mind for this notion, I shall observe, that projections have been used for various purposes in mathematics, and that, when pictures of real objects are drawn, according to the rules of perspective, upon a plane, the distances of the pictures of the different objects from each other will be certain functions of the distances of the different real objects; and in many cases, from the position of the pictures we may determine the position of the objects themselves with advantage. Instances of this are not uncommon in the use of the different projections of the sphere, and, from what we have already said of functions, the Reader will see that the determination of the points in the picture which are the representations of real points without, is only the determination of different points in a plane, whose distances from certain given points in the same plane are expressible by certain functions of certain data. If [*Fig.* 5] E be the eye, EB the principal ray cutting the picture in B, BC a line in the plane of the picture, D the object in the plane of EBC, whose representation is required, DC a right line perp. BC, cutting it in C; let ED joined cut BC in P, then it is plain and well known that P will be the representation to the eye at E of the point D, and PC the representation of CD, which is perpendicular to CP; and thus, without going into any metaphysical ideas, are we reminded that a point in a right line may be the representation of a point out of it; and if EB $=a$, DC$=x$, and CB$=b$, we have BP$=\frac{ab}{a+x}$; that is, BP is a function of DC or x, which is represented by $\frac{ab}{a+x}$; and if, instead of projecting by right lines, of which EPD is one, we had projected by curve lines, we should then have had DP a different function of DC; and thus the Reader may understand the notion of functional projection, a term which I here venture to introduce into mathematics. Mercator's projection of maps is another particular case of functional projection, as is the Globular projection, or any other projection projected according to a particular function.

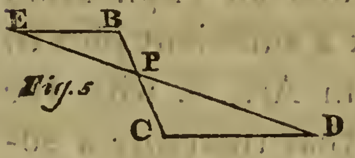


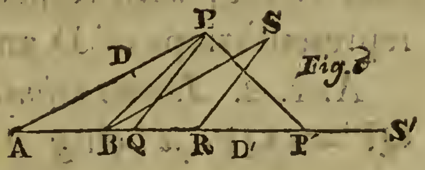
Fig. 5

BOOK II.

ON IMAGINARY QUANTITIES.

ARTICLE I.

GENERAL THEOREM.



If P *be a point out of a right line* AB, *and* Q *be a given point therein, and the angle* AQP *be likewise given, then if* P′ *be taken in the right line with* A *and* B, *so that* QP′ *may be a given function of* QP, *then shall* AP′ *be a determinable function of* AP *and angle* A [Fig. 6].

FOR PQ is a determinable function of (AP, angle A, and angle Q), or, the angle Q being given, a determinable function of AP and the angle A; but QP′ is a given function of QP; it is, therefore, a determinable function of AP and the angle A. Again, AQ being a determinable function of AP and the angles A and Q, or the angle Q being given, a determinable function of AP and the angle A; consequently AP′ is a determinable function of AP and the angle A, it being the sum of two determinable functions of AP and the angle A, of which the one is AQ and the other QP. Q.E.D.

Corollary 1.—BP′ is the same function of PB and the angle B, that AP′ is of AP and the angle A.

Definition 1.—P′ is said to be the representation of P.

Definition 2.—If the angle SRS′ = the angle PQP′, RS′ the same function of RS that P′Q is of PQ, then is S′ said to be a similar representation of S in R′S′ that P′ is of P in QP′, and *vice versâ*.

Corollary 2.—Wherever A and B be taken in the right line AB, BS' is the same function of BS and the angle SBS', that AP' is of AP and the angle A.

Definition 3.—BS' is said to be the measure of BS, and AP' the measure of AP in the right line AB; and the character of the function of PQ, which is the value of PQ, is said to be the character of that measure, Q being a given or constant angle; and that character is likewise said to be the character of representation.

Corollary 3.—The measure of AP, on the line AB, is equal to AB added to the measure of PB on that line, if both measures have the same character.

Definition 4.—$D'P'$, or its equal, is said to be the *directional magnitude* of DP, with respect to the right line AB, if D' and P' be similar representations of D and P in AB.

ART. 2. *Illustration.*—Suppose PQ perpendicular AQ [*Fig.* 7], and $QP'=a.QP$ (a being a given quantity); $PQ=AP.F$ (angle PAQ), and $AQ=AP.f$ (angle PAQ), F and f being the characteristics of given, or at least determinable, functions; then shall $AP'=AP.f(\text{angle PAQ})+a.AP.F$ (angle PAQ). This may be represented by $AP.\phi$ (angle PAQ), in which ϕ stands for the characteristic of a function, which, because a is given, is itself determinable. Let G be some other point in the plane PAQ, and P' the similar representation, in the right line AQP' of G, that it is of P; then, by definition and notation, $AP'=AG.\phi$ (angle GAQ). Draw PK parallel to AQ, cutting the right line AG in K; draw, likewise, Gn, Gm perpendicular to PK, PA, cutting them in n and m. Also demit KS perpendicular AQ, cutting it in S; and let K' be the similar representation of K in AQ that P' is of P; that is, let $SK'=a.KS$ $(=a.PQ=)$ QP', because $PQ=KS$; therefore SQ, or its equal KP, is $=K'P'$; but, by hypothesis and Corollary 2, AP' is $=AG.\phi$ (angle GAQ) $=GK\phi$ (angle GKP) $+$ KA.ϕ (angle KAS); but, by definition, $AK'=AK.\phi$ angle KAS; therefore $AP'=GK\,(\phi GKP)+AK$, therefore $K'P'$, or its equal

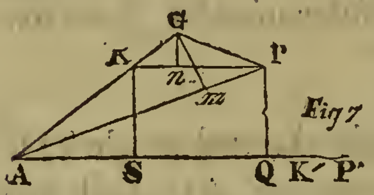
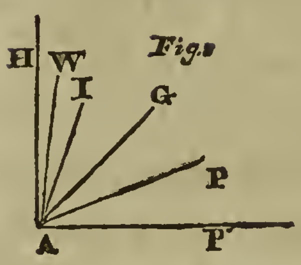


Fig 7.

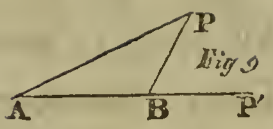
KP, is $=\text{GK}.\phi$ (angle GKP); and consequently, by definition, P is a similar representation of G in the right line KP, that P′ is of P in the right line AQ, and therefore $\text{P}m=a.\text{G}m$. Now, if it were further required that P should likewise be the similar representation of G in AP, we should have $n\text{P}=a.\text{G}n$, and consequently $(\text{GP}^2=)\ \text{G}m^2+a^2.\text{G}m^2=\text{G}n^2+a^2.\text{G}n^2$, that is, $\overline{1+a^2}.\text{G}m^2=\overline{1+a^2}.\text{G}n^2$. This equation is arithmetically impossible, unless $\text{G}n^2$ be equal to $\text{G}m^2$, because the factor $1+a^2$ cannot be equal to 0; but if we imagine that $1+a^2$ can be equal to 0, then we might imagine that the equation could subsist for all the values of Gm and Gn, and, the requisite of $1+a^2$ being $=0$, is, that $a=\sqrt{-1}$. This assumption, notwithstanding its impossibility, affords curious analytical inferences; and though, arithmetically, the assumption may appear to have no meaning, I trust I shall be able to shew both its use in geometry and its meaning in arithmetic. The representation which this gives is not real but imaginary: the character of this *imaginary* representation is, that PQP′ being a right angle, and P′ the representation of P, that QP′ is always $=\sqrt{-1}.\text{QP}$; and, from the above consideration, if G has likewise its similar representation in AQ at the same point P′, P will be the similar representation of G in the line AP; it likewise follows that $\text{AP}'=\text{AP}.\phi(\angle\text{PAP}')$, and $=\text{AG}\phi(\angle\text{GAQ})$, and also that $\text{AP}=\text{AG}\phi(\angle\text{GAP})$, ϕ being the same functional characteristic throughout. Here the representation in AQ, whether considered the result of successive or of immediate representation, is the same point P, whatever the lines may be through which the successive representations are carried.

Corollary:—The directional magnitude of any right line (in this representation) with respect to two right lines parallel to each other, are equal. For KP [*Fig.* 7] is the directional magnitude of GK with regard to the right line passing through K and P, and K′P′ is the directional magnitude of GK with regard to AQ, which is parallel to KP, and KP is =K′P′.



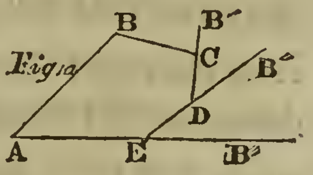
Art. 3. And so [*Fig.* 8] if HA be perpendicular AP′, and we divide the angle HAP′ into any number of equal parts or angles HAI, IAG, GAP, PAP′, by means of the right lines AI, AG, AP; and P′ (whose place is imaginary), be the representation of the last mentioned character of P, G, I, H in the line AP′, we shall have $AP'=AP.\varphi(\angle PAP')$, $AP=AG.\varphi(\angle GAP)=AG.\varphi(\angle PAP')$ $AG=AI.\varphi(\angle IAG)=AI.\varphi(\angle PAP')$ &c.; therefore $AP'=AP.\varphi(\angle PAP')=AG.(\varphi(\angle PAP'))^2=AI.(\varphi(\angle PAP'))^3=$ &c.; and therefore if n were a whole number and the angle $PAP'=\frac{\angle HAP}{n}$, $\angle HAP$ being a right angle, we should have $AP'=AH.\overline{-1}|^{\frac{1}{2}}$, and also $=AH.\overline{\varphi(\angle PAP)}|^{n}$; therefore $\varphi(\angle PAP)=\overline{-1}|^{\frac{1}{2n}}$, and therefore $AP'=AP.\overline{-1}|^{\frac{1}{2n}}=AG.\overline{-1}|^{\frac{2}{2n}}=AI.\overline{-1}|^{\frac{3}{2n}}=$ &c.; and if the angle $WAP'=m\times$ angle PAP', or, which is the same thing, $\frac{m}{n}$. angle HAP′, n and m being whole numbers, and P′ be still the similar representation of W as of P, we shall have $AP'=AW.\overline{-1}|^{\frac{m}{2n}}=AW.\overline{-1}|^{\frac{\angle WAP'}{2}}$, the angle HAP′ being denoted by unity, or equal to $AW\overline{-1}|^{\angle WAP'}$, if the angle HAP be denoted by $\frac{1}{2}$, or twice the angle HAP′, that is, two right angles be denoted by unity, whether WAP′ be a divisor of HAP′ or not; and because we shall have HA in the same position if the angle P′AH be taken $=\overline{1+4\pi}$ right angles, π being either 0 or a whole number positive or negative, therefore AP′ will be equal to $AW.\overline{-1}|^{\angle WAP'}$, if $2+8\pi$ right angles be denoted by unity, π being either 0 or a whole number positive or negative.

[*Fig.* 9.] Still reserving the same character of representation, if P′ be



the representation of P in the right line AB, then AP′ being =AB+BP′, it follows that, if two right angles be denoted by unity, the angle PAB by n, and the angle PBP′ by m (since $AP'=AP\,.\,\overline{-1}|^{n}$, and $BP'=BP\,.\,\overline{-1}|^{m}$), we have $(AP'=)\ AP\,.\,\overline{-1}|^{n}=AB+\overline{-1}|^{m}\,.\,PB$, a *remarkable and useful property*. As an example of the truth and use of this property, let us suppose that the angle B, which is represented by m, is a right angle, and that two right angles be denoted by unity, that is, take $m=\frac{1}{2}$; consequently $(AP'=)AP\,.\,\overline{-1}|^{n}=AB+\overline{-1}|^{\frac{1}{2}}\,.\,PB$. Now, let the angle $n=45°$, that is, let $n=\frac{1}{4}$; therefore we have $AP\,.\,\overline{-1}|^{\frac{1}{4}}=\overline{1+\overline{-1}|^{\frac{1}{2}}}\,.\,PB$; because PB=AB in this case, and because $PB=\sqrt{\tfrac{1}{2}}\,.\,AP$, this equation will become $\overline{-1}|^{\frac{1}{4}}=(1+\overline{-1}|^{\frac{1}{2}})\,.\sqrt{\tfrac{1}{2}}$. By squaring both sides of this equation we have $\overline{-1}|^{\frac{1}{2}}=(1+2\,.\,\overline{-1}|^{\frac{1}{2}}-1)\times\frac{1}{2}$, or its equal $\overline{-1}|^{\frac{1}{2}}$, which, being identical, requires no further proof. Again, m being still $\frac{1}{2}$, or a right angle, let n correspond to 60° or be equal $\frac{1}{3}$, therefore $AP\,.\,\overline{-1}|^{\frac{1}{3}}=AB+BP\,.\,\overline{-1}|^{\frac{1}{2}}$; but if angle A=60°, and B= a right angle, AB will $=\frac{1}{2}AP$, and $PB=\sqrt{\tfrac{3}{4}}.AP$; consequently the equation becomes $\overline{-1}|^{\frac{1}{3}}=(\frac{1}{2}+\sqrt{\tfrac{3}{4}}\,.\,\overline{-1}|^{\frac{1}{2}}=)\frac{1+\sqrt{-3}}{2}$.

Cube both sides, and we have $1=(\frac{1+3\sqrt{-3}-9-3\sqrt{-3}}{8}=)-1$, an equation of identity.



*Art. 4. And if there be ever so many points in a plane [*Fig.* 10], five for instance, A, E, D, C, B, let B′ be the representation of B in CD, B″ the representation of B in ED, and lastly, B‴ the representation of B″ in AE, the representation being still imaginary, and of the last character, and we shall have (Art. 2) B‴ the representation of B in AE, and consequently $AB'''=AB\,.\,\overline{-1}|^{\angle BAE}$, two

right angles being denoted by unity (or $2+8_{\pi}$ right angles being denoted by unity); also $B'C = BC . \overline{-1}\rceil^{\angle BCB'}$; therefore $B'D = BC . \overline{-1}\rceil^{\angle BCB'} + CD$; therefore $DB''' = (B'D . \overline{-1}\rceil^{\angle B'DB''} =)\ BC . \overline{-1}\rceil^{\angle BCB' + \angle B'DB''} + CD . \overline{-1}\rceil^{\angle B'DB''}$; therefore $EB'' = BC . \overline{-1}\rceil^{\angle BCB' + \angle B'DB''} + CD . \overline{-1}\rceil^{\angle B'DB'} + ED$; therefore EB''' or $EB'' . \overline{-1}\rceil^{\angle B''EB'''} = BC . \overline{-1}\rceil^{\angle BCE' + \angle B'DB'' + \angle B''EB'''} + CD . \overline{-1}\rceil^{\angle B'DB'' + \angle B''EB'''} + ED . \overline{-1}\rceil^{\angle B''EB'''}$; and consequently AB''' or $AB . \overline{-1}\rceil^{\angle BAE} = AE + ED . \overline{-1}\rceil^{\angle B''EB'''} + CD . \overline{-1}\rceil^{\angle B''EB''' + \angle B'DB''} + BC . \overline{-1}\rceil^{\angle B''EB''' + \angle B'DB'' + \angle BCB'}$.

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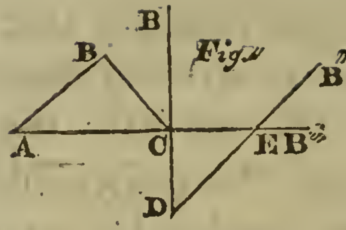
* We remind the Reader that there are no places for the points B′, B″, B‴, their situation being imaginary.

Here we do not conceive it improper to observe, that the imaginary representation B''' may be considered as an analytical representation of the point B with regard to any given line AE, since the expression of AB''' not only points out distinctly the magnitude of AB, or the distance the point B is from A, but it likewise shews the direction in which it lies from A; for AB''' being $= AB . \overline{-1}\rceil^{\angle BAB''}$, it consists of two distinct parts, of which one is the magnitude (AB), and the other the angle (BAB'''); but what seems still more striking in this, I believe, new expression, is, that it contains *every possible path* in the same plane by which we may arrive from A to B; thus you may go from A to B by first going from A to E, then from E to D, then from D to C, and, lastly, from C to B; and the analytical developement of the expression for AB'', namely $AB . \overline{-1}\rceil^{\angle BAE}$, which was above shewn to be $AE + ED . \overline{-1}\rceil^{\angle B''EB'''} + DC . \overline{-1}\rceil^{\angle B''EB''' + \angle B'DB''} + BC . \overline{-1}\rceil^{B''EB''' + B'DB'' + BCB'}$, contains that very fact. The first term denotes the distance from A to E in the right line AE; the part $ED . \overline{-1}\rceil^{\angle B''EB'''}$ shews that D is at the distance ED from E, and in the direction which makes, with EB''', the angle $B''EB'''$; the part $DC . \overline{-1}\rceil^{\angle B''EB''' + \angle B'DB''}$ shews that C lies at the distance CD from D, and in the direction which makes with DB'' the angle $B'DB''$, or which makes with EB''' the angle equal to $\angle B''EB''' + \angle B'DB''$, &c.

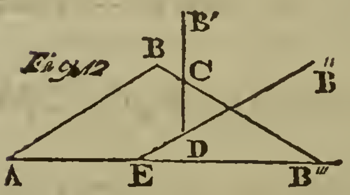
Anxious that the Reader shall perfectly understand the meaning of what I have just asserted, I shall propose, for a particular example, that AB is equal to $\sqrt{2}$, and that the path we wish to take in arriving from A to B is by going over four straight lines AE, ED, DC, and BC; the angle BAE being 45°, and we have $AB''' = AB \,.\, \overline{-1}|^{\angle BAE} = \sqrt{2} \times \overline{-1}|^{\frac{1}{4}}$. Now I observe that $\sqrt{2} \times \overline{-1}|^{\frac{1}{4}}$ may be shewn to be developable into the expression $1 + \frac{1}{3}\sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}} + \frac{1}{3} \,.\, \overline{-1}|^{\frac{1}{4}+\frac{1}{4}} + \frac{1}{3} \,.\, \sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}$, or its equal $1 + \frac{1}{3}\sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}} + \frac{1}{3} \,.\, \overline{-1}|^{\frac{1}{2}} + \frac{1}{3}\sqrt{2} \,.\, \overline{-1}|^{\frac{3}{4}}$, for we have shewn that $\overline{-1}|^{\frac{1}{4}} = \frac{1+\sqrt{-1}}{\sqrt{2}}$, and because $\overline{-1}|^{\frac{3}{4}}$ is $= \overline{-1}|^{\frac{1}{4}} \times \overline{-1}|^{\frac{1}{2}}$, we have $\overline{-1}|^{\frac{3}{4}} = \frac{1+\sqrt{-1}}{\sqrt{2}} \times \sqrt{-1} = \frac{\sqrt{-1}-1}{\sqrt{2}}$; whence the asserted equality of $\sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}}$ and $1 + \frac{1}{3}\sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}} + \frac{1}{3} \,.\, \overline{-1}|^{\frac{1}{2}} + \frac{1}{3}\sqrt{2} \,.\, \overline{-1}|^{\frac{3}{4}}$ is evidently true, the first being equal $1+\sqrt{-1}$, and the second $= 1 + \frac{1+\sqrt{-1}}{3} + \frac{\sqrt{-1}}{3} + \frac{\sqrt{-1}-1}{3}$, and therefore $= 1+\sqrt{-1}$. Hence, in virtue of the theorem, we have $AB \,.\, \overline{-1}|^{\frac{1}{4}} = AE + ED \,.\, \overline{-1}|^{\frac{1}{4}} + DC \,.\, \overline{-1}|^{\frac{1}{4}+\frac{1}{4}} + BC \,.\, \overline{-1}|^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}$, AB, AE, ED, DC, and BC, being respectively equal to $\sqrt{2}$, 1, $\frac{1}{3}\sqrt{2}$, $\frac{1}{3}$, and $\frac{1}{3}\sqrt{2}$; and the angles BAE, DEB‴, CDB″, BCB′ being each $\frac{1}{4}$ of two right angles, that is, 45°; and the truth of this will easily appear by common geometry.

And in a similar manner deductions may be made from every one of the infinite developements, or manners of expressing $\sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}}$; but I shall only add another example, deduced from the developement of $\sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}}$ into the form $3 - \sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}} + \overline{-1}|^{\frac{1}{4}+\frac{1}{4}} + \sqrt{2} \,.\, \overline{-1}|^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}$, an equality which is evident in consequence of $\overline{-1}|^{\frac{1}{4}}$ being $= \frac{1+\overline{-1}|^{\frac{1}{2}}}{\sqrt{2}}$, $\overline{-1}|^{\frac{1}{4}+\frac{1}{4}} = \overline{-1}|^{\frac{1}{2}}$ and $\overline{-1}|^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \overline{-1}|^{\frac{1}{2}+\frac{1}{4}} = \frac{\overline{-1}|^{\frac{1}{2}}-1}{\sqrt{2}}$; and I say, from our theorem, taking $AB=2$, $AE=3$, $DE=-\sqrt{2}$, $DC=1$, $CB=\sqrt{2}$, we may take the

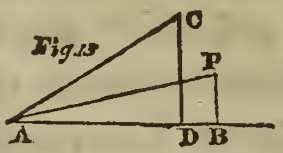


angles BAB‴, B″EB‴, B′DB‴, BCB′ each equal to 45° or $\frac{1}{4}$; and therefore, in *Fig*. 11, we may arrive from A to B by going from A to E, from E to D, from D to C, and from C to B, the angles BAE, B″EB‴, B″DB′, and BCB′ being each 45°, and AB, DE, BC, being each $=\sqrt{2}$, AE$=3$, and DC$=1$; and the truth of this is quite plain from common geometry.

Corollary:—In this representation it is evident that [*Fig*. 12], since if BC, CD, be continued to cut AE, the angles of intersection of those lines with AE will be respectively BCB′ + B′DB″ + B″EB‴, and B′DB″+ B″EB‴. The equation, by corollary to Art. 2, will stand thus; the directional magnitude of BA $=$ AE+ the directional magnitude of ED + the directional magnitude of BC; the directional magnitudes being all taken with respect to the line AE. And a similar theorem will exist for any number of lines.



Art. 5. Let two right angles, or $2+8\pi$ right angles, be still denoted by unity; let also the two right lines AP and AC be equal to each other, and their magnitude be represented by r; let angle PAD$=n$, angle CAD$=mn$, and the angles ADC, ABP be each right angles; then ADB, DC, and PB, being right lines, by the definition of sines and cosines PB is the sine of n, AB the cosine of n, CD the sine of nm, AD the cosine of nm, all to the radius r; but, by Art. 3, AC . $\overline{-1|}^{n\cdot m}=$ AD$+\overline{-1|}^{\frac{1}{2}}$.DC, and AP . $\overline{-1|}^{n}=$ AB $+\overline{-1|}^{\frac{1}{2}}$.PB; that is, $r\,.\,\overline{-1|}^{m\cdot n}=$ cos. of $m.n+\overline{-1|}^{\frac{1}{2}}$. sine of $m.n$, and $r\,.\,\overline{-1|}^{n}=$ cos. of $n+\overline{-1|}^{\frac{1}{2}}$. sine of n; and consequently, from these two, we get $\frac{\text{cos.of } m.n+\overline{-1|}^{\frac{1}{2}}.\text{ sine of } m.n}{r}=\overline{\frac{\text{cos.of } n+\overline{-1|}^{\frac{1}{2}}.\text{ sine of } n}{r}\Big|}^{m}$. I consider it my duty to recommend the Reader to pay particular attention to



this, I believe, new mode of arriving at this very beautiful theorem of Demoivre.

The theorem $r\,.\,\overline{-1}|^{n} =$ cos. of $\bar{n} + \overline{-1}|^{\frac{1}{2}}$. sine of $\bar{n}$ is similar to that on p. 139 Leçons sur le Calcul des Fonctions, &c. published in the year 1806, of La Grange, though derived from very different principles, except that there one right angle instead of two is denoted by unity, and radius is here denoted by unity.

Corollary 1.—We observe that Article 3 likewise gives (two right angles, &c. being denoted by unity) $\mathrm{PA}\,.\,\overline{-1}|^{\angle \mathrm{APB}} = \mathrm{PB} + \overline{-1}|^{\frac{1}{2}}\,.\,\mathrm{AB}$, that is, $r\,.\,\overline{-1}|^{\frac{1}{2}-n} = \mathrm{PB} + \overline{-1}|^{\frac{1}{2}}\,.\,\mathrm{AB}$; or, dividing by $\overline{-1}|^{\frac{1}{2}}$, $r\,.\,\overline{-1}|^{-n} = \dfrac{\mathrm{PB}}{\overline{-1}|^{\frac{1}{2}}} + \mathrm{AB} = \mathrm{AB} - \overline{-1}|^{\frac{1}{2}}\,\mathrm{PB} =$ cos. of $n - \overline{-1}|^{\frac{1}{2}}$. sine of n. Hence we observe, that the two equations $r\,.\,\overline{-1}|^{n} =$ cos. of $n + \overline{-1}|^{\frac{1}{2}}$ sine of n, and $r\,.\,\overline{-1}|^{-n} =$ cos. of $n - \overline{-1}|^{\frac{1}{2}}$. sine of n take place at the same time; add them together and divide the sum by 2, and we get cos. of $n = \frac{1}{2} r\,.\,(\overline{-1}|^{n} + \overline{-1}|^{-n})$; subtract one from the other, and divide by $2\sqrt{-1}$, and we get sine of $n = r\,.\,\dfrac{\overline{-1}|^{n} - \overline{-1}|^{-n}}{2\,.\,\overline{-1}|^{\frac{1}{2}}}$. I mention, by the bye, that from the two equations $\mathrm{AP}\,.\,\overline{-1}|^{n} = \mathrm{AB} + \overline{-1}|^{\frac{1}{2}}\,.\,\mathrm{PB}$, and $\mathrm{AP}\,.\,\overline{-1}|^{-n} = \mathrm{AB} - \overline{-1}|^{\frac{1}{2}}\,.\,\mathrm{PB}$, by multiplication we have $\mathrm{AP}^2 = \mathrm{AB}^2 + \mathrm{PB}^2$. It may appear singular that the imaginary projection should point out so simple a theorem as that of 47 Euc. 1, but the Reader may also observe that I have deduced the theory of this projection from that theorem.

Corollary 2.— Because cos. of $n = \frac{1}{2} r\,.\,(\overline{-1}|^{n} + \overline{-1}|^{-n})$ sine of $n = \dfrac{\overline{-1}|^{n} - \overline{-1}|^{-n}}{2\,.\,\overline{-1}|^{\frac{1}{2}}}$ we have tangent of $n = \dfrac{r\,.\,\text{sine of } n}{\text{cos. of } n} = \dfrac{\overline{-1}|^{n} - \overline{-1}|^{-n}}{\overline{-1}|^{\frac{1}{2}}.(\overline{-1}|^{n} + \overline{-1}|^{-n})}$

$=\dfrac{\overline{-1|}^{2n}-1}{\overline{-1|}^{\frac{1}{2}}.(\overline{-1|}^{2n}+1)}$; and by resolution we get $\overline{-1|}^{2n}=$ $\dfrac{1+\overline{-1|}^{\frac{1}{2}}.\text{tangent of } n}{1-\overline{-1|}^{\frac{1}{2}}.\text{tangent of } n}$.

Corollary 3.—If the angle PAB be extremely small, AB may be considered not to differ from AP in magnitude; and therefore, if AP=1, we shall have the equation $\overline{-1|}^{n}=1+\overline{-1|}^{\frac{1}{2}}.\text{PB}$; but because angle PAB is extremely small, PB be considered the same as the arc of a circle described about A, with the radius AP, and intercepted by the radii AP, AB, and consequently if c be the semi-circumference of the circle, we have PB$=n.c$, two right angles being denoted by unity; therefore we have $\overline{-1|}^{n}=1+\overline{-1|}^{\frac{1}{2}}.nc$; and therefore $c=\dfrac{\overline{-1|}^{n}-1}{n.\overline{-1|}^{\frac{1}{2}}}$, n being infinitely small.

Corollary 4.—If n be infinitely small, it is well known that the hyperbolical logarithm of p is $=\dfrac{p^{n}-1}{n}$; consequently, writing -1 in the room of p in Corollary 3, we have $c=\dfrac{\text{hyp.log.of }(-1)}{\overline{-1|}^{\frac{1}{2}}}$. Hence if ε be put for the number whose hyperbolical logarithm is 1, because from the last equation we have $\overline{-1|}^{\frac{1}{2}}.c=\text{hyp.log. of }(-1)$, we have $\varepsilon^{c.\overline{-1|}^{\frac{1}{2}}}=-1$.

Corollary 5.—Hence, writing $\varepsilon^{c.\overline{-1|}^{\frac{1}{2}}}$ instead of -1 in the theorems of Corollary 1, we get (whatever n is, the circular arc corresponding to the rad. r, and the $\angle\, n$ being $=z$) the well known theorems $r.\varepsilon^{\frac{z}{r}\sqrt{-1}}=\cos.\text{of } z$

$+\sqrt{-1}$. sine of z, cos. of $z=\frac{1}{2}r.(\varepsilon^{\frac{z}{r}\sqrt{-1}}+\varepsilon^{-\frac{z}{r}\sqrt{-1}})$; and sine of $z=\frac{\frac{1}{2}r.}{\sqrt{-1}}.(\varepsilon^{\frac{z}{r}\sqrt{-1}}-\varepsilon^{-\frac{z}{r}\sqrt{-1}})$.

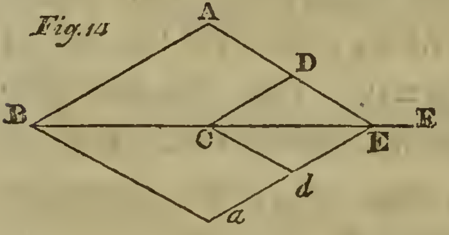
Art. 6.—No. 1. Hence if there be a given equation, $a.\overline{-1|}^{b}+c.\overline{-1|}^{d}+x.\overline{-1|}^{y}=g$, in which a, b, c, d, and g, are given real quantities, and x and y real, we have the means of determining both x and sine of y. For, denoting two right angles by unity, we have, from Article 5, $\overline{-1|}^{b}=$ cos. of $b+\sqrt{-1}$. sine of b, $\overline{-1|}^{c}=$ cos. of $c+\sqrt{-1}$. sine of c, and $\overline{-1|}^{y}=$ cos. of $y+\sqrt{-1}$. sine of y; consequently, these values being put in the proposed equation, we obtain a. cos. of $b+c$ cos. of $d+x$ cos. of $y+\sqrt{-1}$ (a sine of $b+c$ sine of $d+x$ sine of y)$=g$; but because all the letters represent real values, the imaginary part must vanish of itself; consequently we get a . cos. of $b+c$ cos. of $d+x$ cos. of $y=g$; and a . sine of $b+c$. sine of $d+x$ sine of $y=0$. Put, for the sake of brevity, $g-a$. cos. of $b-c$ cos. of $d=\mathrm{A}$, and a . sine of $b+c$. sine of $d=\mathrm{B}$; therefore we have x . cos. of $y=\mathrm{A}$, and x sine of $y=\mathrm{B}$; therefore $\frac{\text{sine of } y}{\text{cos. of } y}$, or (radius being unity) tangent of $y=-\frac{\mathrm{B}}{\mathrm{A}}$, and $y=$ angle whose tangent is $-\frac{\mathrm{B}}{\mathrm{A}}$, and $x=\frac{\mathrm{B}}{\text{sine of } y}=\sqrt{\mathrm{A}^2+\mathrm{B}^2}$. Hence we see that our equation containing two unknown quantities x and y, does, of itself, determine x and the sine of y! And the same would follow if there were more of the given terms a, $\overline{-1|}^{b}$, &c. in the equation.

No. 2. If a and c were the unknown values in the equation $a.\overline{-1|}^{b}+c.\overline{-1|}^{d}+x.\overline{-1|}^{y}=g$, we have, by multiplying the equation, a . cos. of $b+c$ cos. of $d+x$. cos. of $y=g$, by the sine of b; and the equation a . sine of $b+c$ sine of $d+x$ sine of $y=0$, by the cos. of b, and, taking the difference of the two resulting equations, c (cos. of d . sine of b — sine of b . cos. of d) $+x$(cos. of y . sine of b — sine of y . cos. of b) $=g$ sine of b; that is, c sine of $(b-d)+x$. sine of $(b-y)=g$. sine of b;

therefore $c = \frac{g\,.\,\text{sine of } b-x\,.\,\text{sine of }\overline{b-y}}{\text{sine of }(b-d)}$. In the same way we get $a = \frac{g\,.\,\text{sine of } d-x\,.\,\text{sine of }\overline{d-y}}{(\text{sine of }\overline{d-b},\ \text{or its equal})-\text{sine }(b-d)}$; and it is not difficult to see how to proceed if any five of the seven quantities a, b, c, d, x, y, and g, are given.

No. 3. The solution of these sort of equations may be obtained differently; for I observe, if the equation $a\,.\,\overline{-1|}^{b}+c\,.\,\overline{-1|}^{d}+x\,.\,\overline{-1|}^{y}=g$, in which a, b, c, d, x, y, and g, are all real, can exist, the equation $a\,.\,\overline{-1|}^{-b}+c\,.\,\overline{-1|}^{-d}+x\,.\,\overline{-1|}^{-y}=g$ is likewise true; for the first of these, by transposition, will stand $x\,.\,\overline{-1|}^{y}=g-a\,.\,\overline{-1|}^{b}-c\,.\,\overline{-1|}^{d}$; and it may be written $x\,.\,\overline{-1|}^{y}=g+a\,.\,\overline{-1|}^{b-1}+c\,.\,\overline{-1|}^{d-1}$; hence, from what has been said [*Fig.* 14], if the right lines AB, BC, CD, AD, be respectively equal to x, g, a, and c; and if AD be continued to cut the right line BCF in E, and two right angles being denoted by unity, the angle DCF be denoted by $b-1$, the angle AEF by $d-1$, then will the angle ABF be represented by x; for we have $\text{AB}\,.\,\overline{-1|}^{\angle\text{ABF}}=\text{BC}+\text{DC}\,.\,\overline{-1|}^{\angle\text{DCF}}+\text{AD}\,.\,\overline{-1|}^{\angle\text{AEF}}$; but if we take B$a$=BA, making the angle aBE, on the contrary side of BE, in magnitude equal to ABE, and construct the figure BCda in every way similar to the figure BCDA, except that it be on the contrary side of BC, we have angle aBC$=-y$, angle dCE$=-b+1$, the angle aEF$=-d+1$; and therefore, in virtue of the equation, $a\text{B}.\overline{-1|}^{\angle a\text{BF}}=\text{BC}+cd.\overline{-1|}^{\angle d\text{CE}}+ad.\overline{-1|}^{\angle a\text{EF}}$; $x.\overline{-1|}^{-y}=g+a\,.\overline{-1|}^{1-b}+c\,.\,\overline{-1|}^{1-d}=g-a.\overline{-1|}^{b}-c\,.\,\overline{-1|}^{-d}$; and therefore $a.\overline{-1|}^{-b}+c\,.\overline{-1|}^{-d}+x.\overline{-1|}^{-y}=g$. Q.E.D. And a similar theorem follows when there are more of these terms.

Fig. 14

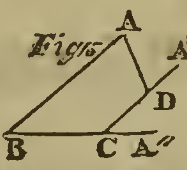


No. 4. That the Reader may the easier perceive the use of this, I shall resume the case of Art. 2, to find a and c in the equation $a.\overline{-1|}^{b}+c.\overline{-1|}^{d}+x.\overline{-1|}^{y}=g$, all the letters but a and c being given; therefore, since by No. 3 we likewise have $a.\overline{-1|}^{-b}+c.\overline{-1|}^{-d}+x.\overline{-1|}^{-y}=g$, I observe that, multiplying the first by $\overline{-1|}^{-b}$ and the second by $\overline{-1|}^{b}$, we have the two equations $a+c.\overline{-1|}^{d-b}+x.\overline{-1|}^{y-b}=g.\overline{-1|}^{-b}$, and $a+c.\overline{-1|}^{-d+b}+x.\overline{-1|}^{-y+b}=g.\overline{-1|}^{b}$; consequently, taking the difference of the two, we have $c.(\overline{-1|}^{d-b}-\overline{-1|}^{-d+b})+x.(\overline{-1|}^{y-b}-\overline{-1|}^{-y+b})=g.(\overline{-1|}^{-b}-\overline{-1|}^{b})$; divide the whole by $2.\overline{-1|}^{\frac{1}{2}}$, and we have, from Cor. 1, Art. 5, $c=\frac{g.\text{sine of } b-x.\text{sine of } (b-y)}{\text{sine of } (b-d)}$, exactly the same as in No. 2; and in the same way is a found.

Scholium:—Before I proceed I shall offer some observations, to obviate what may appear to some Readers a difficulty. I frequently use the expression $2+8\pi$ right angles being denoted by unity, π being either equal to 0, or a whole number positive or negative; and for that purpose I shall propose for consideration the particular case of the equation (radius being taken unity of Cor. 1, Art. 5) $\overline{-1|}^{n}=\text{cos. of } n+\overline{-1|}^{\frac{1}{2}}\text{sine of } n$, $2+8\pi$ right angles being denoted by unity, when $n=\frac{1}{5}$, and we shall have $\overline{-1|}^{\frac{1}{5}}=\text{cos. of } \frac{1}{5}, +\overline{-1|}^{\frac{1}{2}}\text{sine of } \frac{1}{5}$. Now, if $\pi=0$, or two right angles be denoted by unity, then will $\frac{1}{5}$ correspond to an angle of $\frac{180}{5}$ degrees, that is, to 36°, and then the interpretation of the equation will be $\overline{-1|}^{\frac{1}{5}} = \text{cos. of } 36°, +\overline{-1|}^{\frac{1}{2}}.\text{sine of } 36°$; and this, in fact, contains two of the imaginary roots of $\overline{-1|}^{\frac{1}{5}}$. If π be taken equal to unity, then will $\frac{1}{5}$ correspond to $\frac{2+8}{5}$ right angles, that is, two right angles; and in that case $\overline{-1}|^{\frac{1}{5}}$ will come out $=-1$; and this is the real root of $\overline{-1}|^{\frac{1}{5}}$. If π be taken $=2$, then will $\frac{1}{5}$ correspond to $\frac{18}{5}$ right angles or to $(4-\frac{2}{5})$ right angles, whose sine and cosine are the same as the sine and cosine of $-\frac{2}{5}$ of a right angle, or of $-36°$; because four right angles correspond to a whole circumference of a circle; and this falls in with the first case of the interpretation or value of $\overline{-1}|^{\frac{1}{5}}$, in consequence of $\overline{-1}|^{\frac{1}{2}}$ enveloping a double sign $\pm$. If π be taken equal to 3, then $\frac{1}{5}$ will correspond to $\frac{26}{5}$ right angles, that is, it will correspond to a whole circumference $+180°-36°$; and in this case $\overline{-1}|^{\frac{1}{5}}$ will come out $-\text{cos. of } 36° + \overline{-1}|^{\frac{1}{2}}.\text{sine of } 36°$, and this gives the two other imaginary values of $\overline{-1}|^{\frac{1}{5}}$; and if we go on by taking π any other whole number, we shall only get one or other of these five values of $\overline{-1}|^{\frac{1}{5}}$. Moreover, we have just seen that $\frac{1}{5}$ will represent an angle of a different number of degrees, varying with the value of π; we now observe that the same number of degrees will have the value of the angle expressed differently also, as π varies; thus, if $\pi=0$, an angle of 36° will be expressed by $\frac{1}{5}$; if $\pi=1$ (or ten right angles be denoted by unity), then an angle of 36° will be expressed by $\frac{36}{900}$, or its equal $\frac{1}{25}$; but if the angle PAB [*Fig.* 13] corresponds to 36°, we shall have, according to the first of these, $AB.\overline{-1}|^{\frac{1}{5}} = AB+PB.\overline{-1}|^{\frac{1}{2}}$; and, according to the second, $AP.\overline{-1}|^{\frac{1}{25}} = AB+PB.\overline{-1}|^{\frac{1}{2}}$; and therefore it might seem that we ought to infer, that $\overline{-1}|^{\frac{1}{5}} = \overline{-1}|^{\frac{1}{25}}$, which is not the case; the fact is, that five of the twenty-five roots of $\overline{-1}|^{\frac{1}{25}}$ are the same as the five roots of $\overline{-1}|^{\frac{1}{5}}$; and that the interpretations of $\overline{-1}|^{\frac{1}{5}}$ and $\overline{-1}|^{\frac{1}{2.5}}$, according to the method in the first part of this Scholium, shews under what circumstances $\overline{-1}|^{\frac{1}{5}}=\overline{-1}|^{\frac{1}{2.5}}$ is true. Many apparent difficulties, in my opinion, may be explained by the introduction of the three terms *entire equality*, *conditional equality*, and *partial equality*. If the three equations $2.(x+3)=3x$, $y-5=1$, and $z^2+6=7z$ be proposed, we shall find x, y, and z each $=6$, and that x and y can only be 6, but that z may also be 1; we should therefore say, that x and y are entirely equal to each other, but that x and z, or y and z, are only partially equal to each other. By conditional equality I mean that which would exist under certain conditions of operation only.

Art. 7. By way of illustrating the utility, and to prove by particular examples the truth of the observation in Art. 4, I shall propose

Problem I.—*In the right lined figure* ABCD [Fig. 15], *let* AB$=4$, BC$=3$, *and* DC$=2$; *also let the angle* ABC $=\frac{1}{4}$ *of two right angles, that is* 45°, *and angle* BCD$=\frac{2}{3}$ *of two right angles; it is required to find* AD *and the angle* D.



Produce BC to A″, and CD to A′; consequently, two right angles being denoted by unity, the angle DCA″ will be equal to $\frac{1}{3}$. Let AD$=x$, and the angle ADA$'=y-\frac{1}{3}$; then, from what is observed in Art. 4, $4.\overline{-1}|^{\frac{1}{4}}=3+2.\overline{-1}|^{\frac{1}{3}}+x.\overline{-1}|^{y}$; therefore, from Art. 6, No. 1, if A be put $=-3+4$ cos. of $\frac{1}{4}-2$cos. of $\frac{1}{3}$, that is, $-3+4$ cos. of 45°-2cos. of 60°, that is, $=-4+2\sqrt{2}$; and B$=-4$ sine of $\frac{1}{4}+2$ sine of $\frac{1}{3}$, or $-2\sqrt{2}+\sqrt{3}$, we shall have AD, or $x=\sqrt{A^2+B^2}=\sqrt{\overline{2.\sqrt{2}-4}|^2+\overline{\sqrt{3}-2\sqrt{2}}|^2}$; and $y=$ angle whose tangent is $-\frac{B}{A}$, from which take $\frac{1}{3}$ of two right angles and there remains the value of the angle ADA′, which was required. And these are the same answers, as may be easily found by dropping perpendiculars from A and D on BC by the common method.

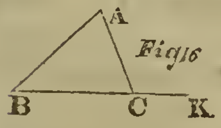
Problem II.—*Let* AB, *angle* B, BC, *angle* C, *and angle* D *be given to find* AD *and* CD.

Put AB$=a$, BC$=b$, CD$=x$, AD$=y$, angle ABC$=n$, angle DCA″$=m$, and angle ADA′$=p$, two right angles being denoted by unity; consequently the equation is $a.\overline{-1|}^{n}=b+x.\overline{-1|}^{m}+y.\overline{-1|}^{m+p}$; and consequently, by Art. 6, No. 3, also $a.\overline{-1|}^{-n}=b+x.\overline{-1|}^{-m}+y.\overline{-1|}^{-m-p}$; multiply the first by $\overline{-1|}^{-m-p}$, and the second by $\overline{-1|}^{m+p}$, and they will become respectively $a.\overline{-1|}^{n-m-p}=b.\overline{-1|}^{-m-p}+x.\overline{-1|}^{-p}+y$, and $a.\overline{-1|}^{m+p-n}=b.\overline{-1|}^{m+p}+x.\overline{-1|}^{p}+y$; hence, taking the difference of the two, &c., we have $x=\dfrac{a.(\overline{-1|}^{m+p-n}-\overline{-1|}^{-m-p+n})-b.(\overline{-1|}^{m+p}-\overline{-1|}^{-m-p})}{\overline{-1|}^{p}-\overline{-1|}^{-p}}=$ (by Art. 5, Cor. 1) $\dfrac{a.\text{sine of }(m+p-n)-b.\text{sine of }(m+p)}{\text{sine of }p}$. And in a similar way we have $y=\dfrac{-a.(\overline{-1|}^{m-n}-\overline{-1|}^{n-m})+b.(\overline{-1|}^{m}-\overline{-1|}^{-m})}{\overline{-1|}^{n}-\overline{-1|}^{-p}}=$

$$\frac{-a.\text{sine of }\overline{m-n}+b.\text{sine of }m}{\text{sine of }p}.$$

Problem III.—*Given two sides* [Fig. 16] AB, BC, *and the including angle* B; *to find the angle* C *and side* AC.

Produce BC to K; let the angle ABC$=n$, two right angles being denoted by unity, and the angle ACK$=p$; consequently (Art. 3) AB$.\overline{-1|}^{n}=$BC$+$AC$.\overline{-1|}^{p}$, or AC$.\overline{-1|}^{p}=$AB$.\overline{-1|}^{n}-$BC; and therefore, also (Art. 6, No. 3), AC$.\overline{-1|}^{-p}=$AB$.\overline{-1|}^{-n}-$BC. Multiplying these two together, we have AC$^2=$AB$^2+$BC$^2-$AB.BC$.(\overline{-1|}^{n}+\overline{-1|}^{-n})$; because $\overline{-1|}^{p}\times\overline{-1|}^{-p}$ is



equal to 1, and also $\overline{-1}|^{n}\times\overline{-1}|^{-n}=1$; therefore AC $=$ $\sqrt{AB^2+BC^2-AB.BC.(\overline{-1}|^{n}+\overline{-1}|^{-n})}$ $=$ (by Article 5, Corollary 1) $\sqrt{AB^2+BC^2-\frac{2AB.BC.\cos.\,(\text{of } n \text{ or of } B)}{\text{radius}}}$.

Again, dividing the equation $AC.\overline{-1}|^{p}=AB.\overline{-1}|^{n}-BC$ by $AC.\overline{-1}|^{-p}=AB.\overline{-1}|^{-n}-BC$, we obtain $\overline{-1}|^{2p}=\dfrac{\overline{-1}|^{n}-\frac{BC}{AB}}{\overline{-1}|^{-n}-\frac{BC}{AB}}$; therefore $\overline{-1}|^{2p}-1=\dfrac{\overline{-1}|^{n}-\overline{-1}|^{-n}}{\overline{-1}|^{-n}-\frac{BC}{AB}}$, and $\overline{-1}|^{2p}+1=\dfrac{\overline{-1}|^{n}+\overline{-1}|^{-n}-\frac{2BC}{AB}}{\overline{-1}|^{-n}-\frac{BC}{AB}}$. Divide the first of the two last of these by the second, multiplied by $\overline{-1}|^{\frac{1}{2}}$, and we have tangent of $p=\dfrac{\overline{-1}|^{n}-\overline{-1}|^{-n}}{\overline{-1}|^{\frac{1}{2}}.(\overline{-1}|^{n}+\overline{-1}|^{-n}-\frac{2BC}{AB})}$; that is, from Cor. 1, Art. 5, tangent of $p=\dfrac{\text{sine of } n}{\cos.\text{of } n-\frac{BC}{AB}}$.

Corollary 1. $\overline{-1}|^{n}+\overline{-1}|^{-n}=\dfrac{AB^2+BC^2-AC^2}{AB.BC}$, or putting k for $\dfrac{AB^2+BC^2-AC^2}{AB.BC}$, $\overline{-1}|^{n}+\overline{-1}|^{-n}=k$; and squaring we have $\overline{-1}|^{2n}+2+\overline{-1}|^{-n}=k^2$; therefore $\overline{-1}|^{2n}-2+\overline{-1}|^{-n}=k^2-4$, and therefore $\overline{-1}|^{n}-\overline{-1}|^{-n}=\sqrt{k^2-4}$; and therefore, because $\overline{-1}|^{n}+\overline{-1}|^{-n}=k$, we have $\overline{-1}|^{n}=\dfrac{k+\sqrt{k^2-4}}{2}$.

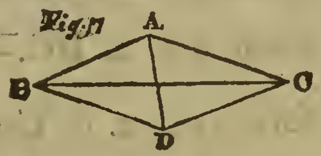
Corollary 2. n logarithm of $(-1) =$ logarithm of $\left(\frac{k}{2} + \sqrt{\frac{1}{4}k^2-1}\right)$ $=$ log. of $\left(\frac{1}{2}k+\sqrt{-1}.\sqrt{1-\frac{1}{4}k^2}\right)$, and $n = \dfrac{\text{log. of } (\frac{1}{2}k+\sqrt{-1}.\sqrt{1-\frac{1}{4}k^2})}{\text{log. of } (-1)}$.

Corollary 3.—Putting $\sqrt{1-\frac{1}{4}k^2} \div \frac{1}{2}k = t$, the value of n in Cor. 2 will become log. of $\frac{1}{2}k\times(1+\sqrt{-1}.t)$ divided by the log. of -1; and, using hyp. logs., we have log. of $(\frac{1}{2}k\times\overline{1+\sqrt{-1}.t})$, or its equal log. of $(\frac{1}{2}k)$ + log. of $(1+\sqrt{-1}.t) =$ log. of $(\frac{1}{2}k)+\sqrt{-1}.t-\frac{(\sqrt{-1}.t)^2}{2}+\frac{(\sqrt{-1}.t)^3}{3}$ + &c. $=$ log. of $(\frac{1}{2}k)+\sqrt{-1}\times\left(t-\frac{t^3}{3}+\frac{t^5}{5}, \text{\&c.}\right)+\frac{t^2}{2}-\frac{t^4}{4}+\frac{t^6}{6}$, &c. $=\sqrt{-1}.\left(t-\frac{t^3}{3}+\frac{t^5}{5}, \text{\&c.}\right)$, barely; because $\frac{t^2}{2}-\frac{t^4}{4}+\frac{t^6}{6}$, &c. $=\frac{1}{2}$ log. of $(1+t^2)=\frac{1}{2}$ log. of $(1+4k^{-2}-1)=\frac{1}{2}$ log. of $(4k^{-2})=-$ log. of $\frac{k}{2}$; hence, c being $=$ the semicircumference of a circle whose radius is unity, since we shall have (by Cor. 4, Art. 5) log. of $(-1)=\sqrt{-1}.c$, we have $n=t-\frac{t^3}{3}+\frac{t^5}{5}$, &c., divided by c.

Problem IV.—*Given angle* B *(see the last* Fig.*), the sum of the sides* AB *and* BC, *and the opposite side* AC; *to find the angle* C.

Two right angles being denoted by unity, put angle ABC$=p$, angle ACK$=q$, AB+BC$=a$, AC$=b$ and BC$=x$; and therefore, because $\text{AB}.\overline{-1}^{\,p}=\text{BC}+\text{AC}.\overline{-1}^{\,q}$, we have $(a-x).\overline{-1}^{\,p}=x+b.\overline{-1}^{\,q}$, and $x=\dfrac{a.\overline{-1}^{\,p}-b.\overline{-1}^{\,q}}{1+\overline{-1}^{\,p}}$, or its equal $\dfrac{a.\overline{-1}^{\,\frac{1}{2}p}-b.\overline{-1}^{\,q-\frac{1}{2}p}}{\overline{-1}^{\,\frac{1}{2}p}+\overline{-1}^{\,-\frac{1}{2}p}}$, and, writing $-p$ and $-q$ for p and q, we have, in virtue of Art. 6, No. 3, also $x=$

$\dfrac{a.\overline{-1}\rceil^{-\frac{1}{2}p}-b.\overline{-1}\rceil^{-q+\frac{1}{2}p}}{\overline{-1}\rceil^{\frac{1}{2}p}+\overline{-1}\rceil^{\frac{1}{2}p}}$; and, consequently, we have $a.\overline{-1}\rceil^{\frac{1}{2}p}-b.\overline{-1}\rceil^{q-\frac{1}{2}p}$ $=a.\overline{-1}\rceil^{-\frac{1}{2}p}-b.\overline{-1}\rceil^{-q+\frac{1}{2}p}$; therefore $\overline{-1}\rceil^{q-\frac{1}{2}p}-\overline{-1}\rceil^{-q+\frac{1}{2}p}=\frac{a}{b}.(\overline{-1}\rceil^{\frac{1}{2}p}$ $-\overline{-1}\rceil^{-\frac{1}{2}p})$, that is, dividing by $2.\overline{-1}\rceil^{\frac{1}{2}}$ (from Cor. 1, Art. 5), sine of $\overline{q-\frac{1}{2}p}=\frac{a}{b}$ sine of $\frac{1}{2}p$, that is, the sine of the excess of the angle ACK above half the angle ABC$=\frac{a}{b}$ sine of ½ angle ABC; whence the angle C is given, &c.



Problem V.—[Fig. 17]. *In the right line figure* ABCD *given* AB, BD, AC, DC, BC, *to calculate* AD.

Put AB$=a$, BD$=b$, CD$=c$, AC$=d$, BC$=e$, angle ABC$=p$, and angle CBD$=q$; put also, for the sake of brevity, $\frac{a^2+e^2-d^2}{ae}=g$, and $\frac{b^2+e^2-c^2}{be}=h$, and we have, by Prob. 3, Art. 7, and its corollary 1, AD $=\sqrt{a^2+b^2-ab\,(\overline{-1}\rceil^{p+q}+\overline{-1}\rceil^{-p-q})}$, $\overline{-1}\rceil^{p}=\frac{g+\sqrt{g^2-4}}{2}$, that is, $\frac{g+\sqrt{4-g^2}.\overline{-1}\rceil^{\frac{1}{2}}}{2}$, and $\overline{-1}\rceil^{q}=\frac{h+\sqrt{4-h^2}.\overline{-1}\rceil^{\frac{1}{2}}}{2}$; therefore $\overline{-1}\rceil^{p+q}=(\overline{-1}\rceil^{p}$ $\times\overline{-1}\rceil^{q}=)\,\frac{gh-\sqrt{4-h^2}.\sqrt{4-g^2}}{4}-\frac{(h.\sqrt{4-g^2}+g\sqrt{4-h^2}).\overline{-1}\rceil^{\frac{1}{2}}}{4}$; and, ∵, by Article 6, No. 3, also $\overline{-1}\rceil^{-p-q}=\frac{gh-\sqrt{4-h^2}\sqrt{4-g^2}}{4}-$

$\frac{h.\sqrt{4-g^2}+g\sqrt{4-h^2}}{4}.\overline{-1}|^{-\frac{1}{2}}$; hence, by addition, because $\overline{-1}|^{-\frac{1}{2}}=\overline{-1}|^{1-\frac{1}{2}}=-\overline{-1}|^{\frac{1}{2}}$, we have $\overline{-1}|^{p+q}+\overline{-1}|^{-p-q}=\frac{gh-\sqrt{4-h^2}.\sqrt{4-g^2}}{2}$; whence, from above, $AD=\sqrt{a^2+b^2-ab.\frac{gh-\sqrt{4-h^2}.\sqrt{4-g^2}}{2}}$.

Problem VI.—(*See the last* Fig.) *Given* AB (=a), *angle* ABC (=*p*, *two right angles being denoted by unity*), BD (=b), *angle* ACB (=n), *and angle* BCD (=m); *to find the rest: being a problem very useful in surveying, and well known.*

Call the angle CBD, x; put AC$=z$ and CD$=y$; whence we have, from Art. 3, $BC.\overline{-1}|^{x}=b+y.\overline{-1}|^{m+x}$, or $BC=b.\overline{-1}|^{-x}+y.\overline{-1}|^{m}$; and therefore, by Art. 6, Cor. 3, BC also$=b.\overline{-1}|^{x}+y.\overline{-1}|^{-m}$; whence, from the two, $y=b.\frac{\overline{-1}|^{x}-\overline{-1}|^{-x}}{\overline{-1}|^{m}-\overline{-1}|^{-m}}$. Moreover, because the supplement of the angle BDC$=x+m$, and the supplement of the angle ACD$=1-n-m$, two right angles being denoted by unity, we have, by Article 4, $a.\overline{-1}|^{p}=b+y.\overline{-1}|^{x+m}+z.\overline{-1}|^{x+m+1-n}$, that is, $a.\overline{-1}|^{p}=b+y.\overline{-1}|^{m+x}-z.\overline{-1}|^{x-n}=$ (by substituting for y) $b+b.\frac{(\overline{-1}|^{m+2x}-\overline{-1}|^{m})}{\overline{-1}|^{m}-\overline{-1}|^{-m}}-z.\overline{-1}|^{x-n}$, that is, $b.\frac{\overline{-1}|^{m+2x}-\overline{-1}|^{-m}}{\overline{-1}|^{m}-\overline{-1}|^{-m}}-z.\overline{-1}|^{x-n}$, and consequently by Article 6, No. 3, also $a.\overline{-1}|^{-p}=b\,\frac{\overline{-1}|^{-m-2x}-\overline{-1}|^{m}}{\overline{-1}|^{-m}-\overline{-1}|^{m}}-z.\overline{-1}|^{-x+n}$; multiply this last by

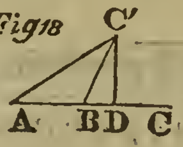
$\overline{-1}|^{2x-2n}$, and it becomes $a.\overline{-1}|^{2x-2n-p}=b.\frac{\overline{-1}|^{-m-2n}-\overline{-1}|^{m+2x-2n}}{\overline{-1}|^{-m}-\overline{-1}|^{m}}$ $-z.\overline{-1}|^{x-n}$; this being subtracted from the equation immediately preceding the last, gives $a.(\overline{-1}|^{p}-\overline{-1}|^{2x-2n-p})$ $=\frac{b.\overline{-1}|^{m+2x}-\overline{-1}|^{-m}+\overline{-1}|^{-m-2n}-\overline{-1}|^{m+2x-2n}}{\overline{-1}|^{m}-\overline{-1}|^{-m}}$; consequently $\overline{-1}|^{2x}=$ $\{a.\overline{-1}|^{p}.(\overline{-1}|^{m}-\overline{-1}|^{-m})-b.\overline{-1}|^{-m}.(\overline{-1}|^{-2n}-1)\}\div\{a.\overline{-1}|^{-2n-p}\times(\overline{-1}|^{m}-$ $\overline{-1}|^{-m})+b.\overline{-1}|^{m}\times(1-\overline{-1}|^{-2n})\}=\frac{\overline{-1}|^{n+p}+k.\overline{-1}|^{-m}}{\overline{-1}|^{-n-p}+k.\overline{-1}|^{m}}$, k being put for $\frac{b}{a}.\frac{\overline{-1}|^{n}-\overline{-1}|^{-n}}{\overline{-1}|^{m}-\overline{-1}|^{-m}}$, or its equal $\frac{b}{a}.\frac{\text{sine of } n}{\text{sine of } m}$; therefore $\overline{-1}|^{2x}-1=$ $\frac{\overline{-1}|^{n+p}-\overline{-1}|^{-n-p}-k.(\overline{-1}|^{m}-\overline{-1}|^{-m})}{\overline{-1}|^{-n-p}+k.\overline{-1}|^{-m}}$; and $\overline{-1}|^{-2x}+1=$ $\frac{\overline{-1}|^{n+p}+\overline{-1}|^{-n-p}+k(.\overline{-1}|^{m}+\overline{-1}|^{m})}{\overline{-1}|^{-n-p}+k.\overline{-1}|^{-m}}$; whence $\frac{\overline{-1}|^{2x}-1}{(\overline{-1}|^{2x}+1).\overline{-1}|^{\frac{1}{2}}}$, or the tangent of $x=\frac{\overline{-1}|^{n+p}-\overline{-1}|^{-n-p}-k.(\overline{-1}|^{m}-\overline{-1}|^{-m})}{\{\overline{-1}|^{n+p}+\overline{-1}|^{-n-p}+k.(\overline{-1}|^{m}+\overline{-1}|^{-m})\}.\overline{-1}|^{\frac{1}{2}}}=\frac{\text{sin.of}\overline{n+p}-k\text{.sin.of}m}{\text{cos.of}\overline{n+p}+k\text{.cos.of}m}$ hence the rest is easily found.

Problem VII.—*In the five sided right lined figure* [*see* Fig. 10] *let* AB, AE, ED, DC *and* BC *be given respectively equal to* a, b, c, d *and* e, *and suppose the angle* DEB″=2. *angle* CDB″, *and the angle* BAE=4. *angle* B′DB″, *it is required to find the rest.*

Put angle CDB″$=x$, and the angle BCB′$=y$, two right angles being denoted by unity, and we have (by Art. 4) $a.\overline{-1}|^{4x}=b+c.\overline{-1}|^{2x}+d.\overline{-1}|^{3x}+e.\overline{-1}|^{y+3x}$, or $e.\overline{-1}|^{y+3x}=-b-c.\overline{-1}|^{2x}-d.\overline{-1}|^{3x}+a.\overline{-1}|^{4x}$; and therefore by Art. 6, No. 3, also $e.\overline{-1}|^{-y-3x}=-b-c.\overline{-1}|^{-2x}-d.\overline{-1}|^{-3x}+a.\overline{-1}|^{-4x}$; hence, multiplying these two together, we have $e^2=b^2+c^2+d^2+a^2+d.(c-a).(\overline{-1}|^{x}+\overline{-1}|^{-x})+c.(b-a).(\overline{-1}|^{2x}+\overline{-1}|^{-2x})+db.(\overline{-1}|^{3x}+\overline{-1}|^{3x})-ab.(\overline{-1}|^{4x}+\overline{-1}|^{-4x})$. And if in this equation we put w for $\overline{-1}|^{x}+\overline{-1}|^{-x}$, that is for twice the cosine of x, we shall convert it into a biquadratic equation, &c.

Art. 8. In the Introduction (Art. 6) there is proposed a question, the equation to which leads to imaginary quantities; and in Art. 7 of the Introduction, I have explained what, according to Monsieur l'Abbé Buée, those imaginary quantities teach: I have there ventured to express my belief, that the theory offered by that gentleman will not hold, and I now propose to shew what may be the meaning of the imaginary roots, and what information they may contain; and, to answer that end, I shall invite the Reader's attention to the solutions I give to the following problems.

Fig18



[Fig. 18.] Problem I.—*The right line* AB *being given in magnitude and position, to find the point* C, *such that* $AC^2=b.BC$, b *being a given magnitude less than* 4AB?

Suppose C to be in a right line with AB continued, put AB$=a$ and BC$=x$, and we have in consequence $\overline{a+x}|^2=bx$; and therefore x or BC $=\tfrac{1}{2}b-a+\sqrt{-1}.\sqrt{b.\overline{a-\tfrac{1}{4}b}}$, and AC$=\tfrac{1}{2}b+\sqrt{-1}.\sqrt{b.\overline{a-\tfrac{1}{4}b}|}$, and these are evidently impossible; the thing being impossible if C be required to be in the line AB continued. But if we take BD in the line AB continued $=\frac{1}{2}b-a$, and CD′ perpendicular to AD and $=\sqrt{b.\overline{a-\frac{1}{4}b}}$, then, because DC is equal to $\sqrt{-1}.\sqrt{b.\overline{a-\frac{1}{4}b}}$, we have $DC=\sqrt{-1}.DC'$; and consequently C is the imaginary representation under consideration of the point C′, and therefore, denoting the angles C′AD, C′BD by A and B, two right angles being denoted by unity, we have, from Art 3 of this work, $BC=BC'.\overline{-1}|^{B}$, and $AC=AC'.\overline{-1}|^{A}$; and therefore, because $AC^2=b.BC$, we have $AC'^2.\overline{-1}|^{2A}=b.BC'.\overline{-1}|^{B}$. Moreover, because $BD=\frac{1}{2}b-a$, and $C'D=\sqrt{b.\overline{a-\frac{1}{4}b}}|$, BC′ is $=a=AB$, and therefore angle BAC′ = angle AC′B, and therefore $2A=B$; whence the equation $AC'^2.\overline{-1}|^{2A}=b.BC'.\overline{-1}|^{B}$ gives $AC'^2=b.BC'$; and therefore the imaginary point C leads to the discovery of a real point C′ having the proposed property; and this is the same point as the Abbé finds. This point can easily be shewn by common geometry to answer; but the method I have here delivered shews the reason of the deduction from the imaginary root, which reason depends on the equality between $\overline{-1}|^{2A}$ and $\overline{-1}|^{B}$.

Problem II.—*It is required to find the point* D, *so that if a given line* AC *be bisected in* B, *we shall have* $AD\times BD\times DC=2AB^3$? [Fig. 19.]

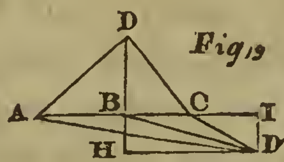
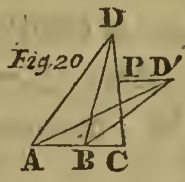


Fig 19

Solution:—If D be assumed in a right line ⊥AC passing through B, and AB=BC be put $=a$, and $DB=x$, we shall have $x.\sqrt{a^2+x^2}|.\sqrt{a^2+x^2}$, or its equal $x^3+a^2x=2a^3$; consequently $x=a$, or $=-\frac{a}{2}\pm\frac{a}{2}\sqrt{-7}$, that is, one root of the equation gives D a real point in the line BD at the distance *a* from B; the other two give D at the imaginary distance in the line BD from $B=-\frac{1}{2}a\pm\frac{a}{2}\sqrt{-7}$. But if DB be produced to H, making $BH=\frac{1}{2}a$, and HD′ be drawn ⊥BH, and equal to $\frac{a}{2}\sqrt{7}$, then because DH is expressed by $\frac{a}{2}.\sqrt{-7}$, we have $DH=D'H.\sqrt{-1}$, and consequently D is the imaginary representation under consideration of the point D′; and consequently if the angle $DAD'=m$, angle $DBD'=n$, and angle $DCD'=p$, two right angles being denoted by unity, we have, from Art. 3, $AD=\overline{-1}|^{m}.AD'$, $BD=\overline{-1}|^{n}.BD'$, and $DC=\overline{-1}|^{p}.D'C$; and consequently because $AD.BD.DC=2AB^3$, we have $AD'.\overline{-1}|^{m}.BD'.\overline{-1}|^{n}.CD'.\overline{-1}|^{p}$, or its equal $AD'.BD'.CD'.\overline{-1}|^{m+n+p}=2AB^3$; but it is evident from our construction that the tangent to radius unity of the angles D′AI, D′BH, and D′CI, are respectively equal to $\frac{1}{\sqrt{7}+2}$, $\sqrt{7}$, and $\frac{1}{\sqrt{7}-2}$, and therefore the tangent of $(D'AI+D'CI)=\sqrt{7}$; and consequently angle D′BH = angle D′AC + angle D′CI; and since angle DAB = angle DCB, consequently angle DAB + angle DCI = two right angles; whence by substitution $m+n+p$ is = two right angles + $HBD+D'BD=4$ right angles $=2$, two right angles having been denoted by unity; and therefore $\overline{-1}|^{m+n+p}=\overline{-1}|^{2}=1$; and therefore $AD'.BD'.CD'=2AB^3$; consequently D′ is a real point answering the conditions of the question, and is a point found from arguing on the imaginary roots yielded by an equation derived from an over-limited presumption. M. l'Abbé Buée would have arrived at the same point from his theory, had he solved this problem; but there are other questions to which his theory would find points which would not answer.

In the preceding solutions we observed that the equations resulting from considering the point to be in a line which the conditions of the question do not admit of, give, as they ought, impossible roots; that is, they give imaginary points for the solution; but these imaginary points were shewn to be imaginary representations of real points according to the functional representation under consideration of real points which answer the conditions of the question. But, if we propose

Problem III.—*In the right line* ABC, BC, AC, *are given in magnitude; and the point* D *is required, so that* AD × BD = c. DC, c *being given.* [Fig. 20.]



Put AC $=a$, BC $=b$ and CD $=x$, CD being perp. to A; and we shall have $\overline{b^2+x^2}\times\overline{a^2+x^2}=c^2x^2$, and therefore, putting $a^2+b^2-c^2=d$, we have $x^4+dx^2=-a^2b^2$, and, in consequence, $x=\sqrt[2]{-\frac{d}{2}+\sqrt[2]{\frac{d^2}{4}-a^2b^2}}$; or, by actual extraction, $x=\frac{1}{2}.\sqrt[2]{2ab-d}+\frac{1}{2}.\sqrt[2]{-1}.\sqrt[2]{2ab+d}$, as evidently will appear by squaring; consequently, if $2ab+d$ be affirmative, this becomes imaginary. Make PC (in the right line DC) $=\frac{1}{2}.\sqrt[2]{2ab-d}$ and PD′ ⊥ DC $=\frac{1}{2}.\sqrt[2]{2ab+d}$; then will the imaginary point D be the imaginary representation under consideration of the real point D′, and consequently AD $=$ AD′ . $\overline{-1}|^{\angle DAD'}$, BD = BD′.$\overline{-1}|^{\angle DBD'}$, and DC = D′C × $\overline{-1}|^{\angle DCD'}$; and consequently, because AD × DB $=c.$DC, we have AD′ .BD′.$\overline{-1}|^{\angle DAD'+\angle DBD'-\angle DCD'}$ $=c.$D′C. Now, if $\overline{-1}|^{\angle DAD'+\angle DBD'-\angle DCD'}$ were equal to unity, we should again have the same property for the real point D′ as is possessed by the imaginary point D; and the theory of M. l'Abbé Buée would teach us to believe, if I mistake not (for he has not solved this problem), that the point D′ would answer the conditions of the problem: but, in consequence of $\overline{-1}|^{\angle DAD'+\angle DBD'-\angle DCD'}$ not being equal to unity, but equal to $\frac{c}{a+b+\sqrt{\overline{a+b}|^2-c^2}}$[*], the case is different, and we are taught that, instead of finding a real point having the property proposed by the question, we have found a point D′ (which is real) such that $AD'.BD'=(a+b+\sqrt{\overline{a+b}|^2-c^2}).CD'$; so that, though the imaginary root does not find a real point having the property proposed, it finds a point whose property it shews; and that the point D′ has that property, is easily proved by common geometry.

ART. 9.—Here it seems proper to repeat an observation in Art. 7 of the Introduction, namely, that the same equation may result from various questions, and I remark that we may always propose a geometrical question, such that the impossible roots of any equation $a+bx+cx^2+dx^3$, &c. $=0$ may be supposed to arise from having represented a certain required point in the question to be placed in a right line, which is inconsistent with the conditions; but such that the impossible roots shall be indicative of imaginary points, significant of the imaginary representation of real points, which do answer the conditions of the question: for this purpose, let the impossible root x be $=p+q\cdot\sqrt{-1}$, and let the equation be resolved into two parts at pleasure, and let it stand, when thus resolved, $A+Bx+Cx^2$&c.$=A'+Bx'+C'x^2$&c. Moreover (which is always possible), let $A+Bx+Cx^2$ &c. be resolved into the form $M.\overline{x+e+e_{,}\sqrt{-1}}^{\,r} \times \overline{x+e-e_{,}\sqrt{-1}}^{\,r} \times \overline{x+f+f_{,}\sqrt{-1}}^{\,s} . \overline{x+f-f_{,}\sqrt{-1}}^{\,s}$. &c.*, and let the part $A'+B'x+C'x^2$ &c., be resolved into the form $M'.\overline{x+e'+e'_{,}\sqrt{-1}}^{\,r'} \times \overline{x+e'-e'_{,}\sqrt{-1}}^{\,r'} \times \overline{x+f'+f'_{,}\sqrt{-1}}^{\,s'} \times \overline{x+f'-f'_{,}\sqrt{-1}}^{\,s'}$, &c., and the equation will become $\overline{x+e+e_{,}\sqrt{-1}}^{\,r} \times \overline{x+e-e_{,}\sqrt{-1}}^{\,r} \times \overline{x+f+f_{,}\sqrt{-1}}^{\,s} \times \overline{x+f-f_{,}\sqrt{-1}}^{\,s}$, &c. $=\frac{M'}{M} \times \overline{x+e'+e'_{,}\sqrt{-1}}^{\,r'} \times \overline{x+e'-e'_{,}\sqrt{-1}}^{\,r'} \times \overline{x+f'+f'_{,}\sqrt{-1}}^{\,s'} \times \overline{x+f'-f'_{,}\sqrt{-1}}^{\,s'}$, &c. Moreover, as x is either equal to $p+q\sqrt{-1}$, or $p-q\sqrt{-1}$, let these be respectively substituted for x in the above equation, and there will result the two following equations, $\overline{p+e+\overline{q+e_{,}}\sqrt{-1}}^{\,r} \times \overline{p+e+\overline{q-e_{,}}\sqrt{-1}}^{\,r} \times \overline{p+f+\overline{q+f_{,}}\sqrt{-1}}^{\,s} \times \overline{p+f+\overline{q-f_{,}}\cdot\sqrt{-1}}^{\,s}$, &c. $=\frac{M'}{M} \times \overline{p+e'+\overline{q+e'_{,}}\sqrt{-1}}^{\,r'} \times \overline{p+e'+\overline{q-e'_{,}}\sqrt{-1}}^{\,r'} \times \overline{p+f'+\overline{q+f'_{,}}\sqrt{-1}}^{\,s'} \times \overline{p+f'+\overline{q-f'_{,}}\sqrt{-1}}^{\,s'}$, &c.; and $\overline{p+e+\overline{e_{,}-q}\sqrt{-1}}^{\,r} \times \overline{p+e+\overline{-q+e_{,}}\sqrt{-1}}^{\,r}$, &c. $=$

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[*] Putting $p=\frac{1}{2}.\sqrt{2ab-d}$, $q=\frac{1}{2}.\sqrt{2ab+d}$, we have $x=p+q\sqrt{-1}$, tangent of DAC $=\frac{p+q\sqrt{-1}}{a}$, tangent of DBC $=\frac{p+q\sqrt{-1}}{b}$, tangent of D′AC $=\frac{p}{a+q}$, tangent of D′BC $=\frac{p}{b+q}$ and tangent of DCP $=\frac{q}{p}$; hence, by Cor. 2, Art. 5, $\overline{-1}|^{2\angle DAC}=\frac{a+p\sqrt{-1}-q}{a-p\sqrt{-1}+q}$, $\overline{-1}|^{2\angle DBC}=\frac{b+p\sqrt{-1}-q}{b-p\sqrt{-1}+q}$, $\overline{-1}|^{2\angle D'AC}=\frac{a+p\sqrt{-1}+q}{a-p\sqrt{-1}+q}$, $\overline{-1}|^{2\angle D'BC}=\frac{b+p\sqrt{-1}+q}{b-p\sqrt{-1}+q}$, and $\overline{-1}|^{2\angle PCD'}=\frac{p+q\sqrt{-1}}{p-q\sqrt{-1}}=\frac{p^2-q^2+2.pq\sqrt{-1}}{p^2+q^2}=\frac{-\frac{1}{2}d+2pq\sqrt{-1}}{ab}$; consequently, because angle DAD′ = angle DAC − angle D′AC, we have $\overline{-1}|^{2\angle DAD'}=(\overline{-1}|^{2\angle DAC-2\angle D'AC}=)$ $\frac{\overline{a^2-p\sqrt{--1}-q}|^2}{\overline{a+q}|^2-\overline{p\sqrt{-1}}|^2}=\frac{a^2+p^2-q^2+2pq\sqrt{-1}}{a^2+2aq+q^2+p^2}=\frac{a^2-\frac{1}{2}d+2pq\sqrt{-1}}{a^2+ab+2aq}$; and, in the same way, $\overline{-1}|^{\angle DBD'}=\frac{b^2-\frac{1}{2}d+2pq\sqrt{-1}}{b^2+ab+2bq}$; therefore, by multiplication, $\overline{-1}|^{2\angle DAD'+2\angle DBD'}=(a^2-\frac{1}{2}d+2pq\sqrt{-1}).(b^2-\frac{1}{2}d+2pq\sqrt{-1})\div ab.(a+b+2q)^2=$, by reduction and restitution, $c^2.(-\frac{1}{2}d+2pq\sqrt{-1})\div ab.(a+b+2pq\sqrt{-1})^2$; whence $\overline{-1}|^{2\angle DAD'+2\angle DBD'-2\angle DCD'}=$, as above said, $c^2\div\overline{a+b+2q}|^2$, that is, $=c^2\div(a+b+\sqrt{\overline{a+b}|^2-c^2}$.

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* M, M′, e, f, $e'_{,}$, f', &c., being independent of x.

$\frac{M'}{M}\times\overline{p+e'+\overline{e'_{,}-q}\sqrt{-1}}^{\,r'}\times\overline{p+e'-\overline{q+e'_{,}}\sqrt{-1}}^{\,r''}$, &c. Multiply the left hand sides of these two equations together, and likewise multiply the right hand sides together, and there will result $\overline{\overline{p+e}^{2}+\overline{q+e_{,}}^{2}}^{\,r}\times\overline{\overline{p+e}^{2}+\overline{q-e_{,}}^{2}}^{\,r}\times$ &c. $=\frac{M'^{2}}{M^{2}}\overline{\overline{p+e'}^{2}+\overline{q+e'_{,}}^{2}}^{\,r'}\times\overline{\overline{p+e'}^{2}+\overline{q-e'_{,}}^{2}}^{\,r'}\times$ &c. And this equation is the equation that would result from the following unlimited question.

Let [*Fig*. 21] a given number of parallel right lines, four for instance, AB, CD, EF, GH, &c. be bisected in I, J, K, L, &c. by the right line IL perpendicular to them. It is required to find the point Z in the same plane; such that $AZ^{r}\times BZ^{r}\times CZ^{s}\times DZ^{s}$ shall be to $EZ^{r'}\times FZ^{r'}\times GZ^{s'}\times HZ^{s'}$ in the given ratio of M′ to M, the points A, B, C, D, E, F, G, H, &c. being given.

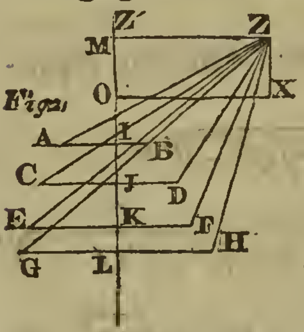
Fig. 21

Solution:—Put $AI=IB=e_{,}$, $CJ=JD=f_{,}$, $EK=KF=e'_{,}$, and $GL=LH=f'_{,}$; also in LI continued take $IO=e$ and let $OJ=f$, $OK=e'$ and $OL=f'$: draw OX parallel to AB, and ZX cutting OX in X perpendicular thereto; let $OX=q$ and $ZX=p$, therefore $ZA=\overline{\overline{p+e}^{2}+\overline{q+e_{,}}^{2}}^{\,\frac{1}{2}}$, $ZB=\overline{\overline{p+e}^{2}+\overline{q-e_{,}}^{2}}^{\,\frac{1}{2}}$, &c., and consequently, by the conditions of the question, we have (as above stated) $\overline{\overline{p+e}^{2}+\overline{q+e_{,}}^{2}}^{\,r}\times\overline{\overline{p+e}^{2}+\overline{q-e_{,}}^{2}}^{\,r}\times$ &c. $=\frac{M'^{2}}{M^{2}}\times\overline{\overline{p+e'}^{2}+\overline{q+e'_{,}}^{2}}^{\,r'}\times\overline{\overline{p+e'}^{2}+\overline{q-e'_{,}}^{2}}^{\,r'}\times$ &c. And this is the general equation to the unlimited problem containing two unknown quantities p and q as it ought. But if, in solving the problem, we had presumed the point in the right line passing through OL at Z′ and OZ′ be put equal to x, it is easy to see, without defacing the diagram with more lines, that we should have had the equation $\overline{\overline{x+e}^{2}+e_{,}^{2}}^{r}\times\overline{\overline{x+f}^{2}+f_{,}^{2}}^{s}=\frac{M'}{M}\times\overline{\overline{x+e'}^{2}+e_{,}'^{2}}^{r'}\times\overline{\overline{x+f'}^{2}+f_{,}'^{2}}^{s'}$, and this will be the same as the equation above, namely, $a+bx+cx^{2}$ &c. $=0$. Now, suppose $x=p+q.\sqrt{-1}$ were an impossible root of this equation, and we should have, at the same time, $\overline{\overline{p+q\sqrt{-1}+e}^{2}+e_{,}^{2}}^{r}\times\overline{\overline{p+q\sqrt{-1}+f}^{2}+f_{,}^{2}}^{s}=\frac{M'}{M}\times\overline{\overline{p+q\sqrt{-1}+e'}^{2}+e_{,}'^{2}}^{r'}\times\overline{\overline{p+q\sqrt{-1}+f'}^{2}+f_{,}'^{2}}^{s'}$ and $\overline{\overline{p-q\sqrt{-1}+e}^{2}+e_{,}^{2}}^{r}\times\overline{\overline{p-q\sqrt{-1}+f}^{2}+f_{,}^{2}}^{s}=\frac{M'}{M}\times\overline{\overline{p-q\sqrt{-1}+e'}^{2}+e_{,}'^{2}}^{r'}\times\overline{\overline{p-q\sqrt{-1}+f'}^{2}+f_{,}'^{2}}^{s'}$. Multiply these two together, the left hand side by the left hand side, and the right hand side by the right hand side, and we shall have, because, in general, $\overline{p+q\sqrt{-1}+e}^{2}+e_{,}^{2}=\overline{p+q\sqrt{-1}+e+e_{,}\sqrt{-1}}\times\overline{p+q\sqrt{-1}+e-e_{,}\sqrt{-1}}$, that is, $\overline{p+e+\sqrt{-1}.(q+e_{,})}\times\overline{p+e+\sqrt{-1}.(q-e_{,})}$; and $\overline{p-q\sqrt{-1}+e}^{2}+e_{,}^{2}=\overline{p+e+\sqrt{-1}.\overline{-q+e_{,}}}\times\overline{p+e-\sqrt{-1}.(q+e_{,})}$; $\overline{\overline{p+e}^{2}+\overline{q+e_{,}}^{2}}^{r}\times\overline{\overline{p+e}^{2}+\overline{q-e_{,}}^{2}}^{r}\times\overline{\overline{p+f}^{2}+\overline{q+f_{,}}^{2}}^{s}\times\overline{\overline{p+f}^{2}+\overline{q-f_{,}}^{2}}^{s}=\frac{M'^{2}}{M^{2}}\times\overline{\overline{p+e'}^{2}+\overline{q+e_{,}'}^{2}}^{r'}\times\overline{\overline{p+e'}^{2}+\overline{q-e_{,}'}^{2}}^{r'}\times\overline{\overline{p+f'}^{2}+\overline{q+f_{,}'}^{2}}^{s'}\times\overline{\overline{p+f'}^{2}+\overline{q-f_{,}'}^{2}}^{s'}$. And consequently it appears, that though the point Z′ in the right line OL, answering the conditions of the problem, is only imaginary, when $x=p+q\sqrt{-1}$, still if we take OM in LO continued, $=p$ and MZ⊥LO$=q$, the point Z′ will be the imaginary representation of the point Z, and will be a solution to the problem. As an example, let it be required to propose a problem requiring the position of a point, which shall be soluble in a particular presumption of the position of a right line, in which the point is to be situate, by the equation given in the Solution to Problem 3, Art. 8, namely, $\overline{b^2+x^2}\times\overline{a^2+x^2}=c^2x^2$, the roots of which are all impossible when c is less than $b+a$; the problem being required to be such, that those impossible roots shall be expressive of the imaginary points in the said assumed right line, which are the imaginary representations of real points answering the conditions of the problem.

Solution:—The equation may be written $\overline{x+b\sqrt{-1}}\times\overline{x-b\sqrt{-1}}\times\overline{x+a\sqrt{-1}}\times\overline{x-a\sqrt{-1}}=c^2x^2$; and consequently, comparing this with the resolved form of the equation preceding, we have $e=o$, $e_{\prime}=b$, $f=o$, $f_{\prime}=a$, $r=1$, $s=1$, $\frac{M'}{M}=c^2$, $e'=o$, $e'_{\prime}=o$, $r'=2$, $s'=o$; therefore AI=IB=b, CJ=JD=a, EK=KF=o, IO=o, OJ=o, and OK=o; and the required problem may be this [*Fig.* 22]. In the right line AB, AI being =IB=b, IC=ID=a, it is required to find the point Z, such that ZA×ZB×ZC×ZD=c^2.ZI^2. Now let the point be presumed in the perpendicular to AB which passes through I, and let IZ=x, and we shall have $\overline{b^2+x^2}.\overline{a^2+x^2}=c^2x^2$, and $x=\frac{1}{2}\sqrt{c^2-\overline{b-a}|^2}+\frac{1}{2}\sqrt{\overline{b+a}|^2-c^2}.\sqrt{-1}$; consequently, if c be less than $b+a$, Z is an imaginary point. In IZ take IP=$\frac{1}{2}.\sqrt{c^2-\overline{b-a}|^2}$, and take PZ′⊥IP and $=\frac{1}{2}.\sqrt{\overline{b+a}|^2-c^2}$, and Z will be the imaginary representation of the real point Z′, which has the required property; and that the point has that property, may be easily proved from common principles.

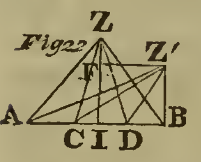
Fig22

ART. 10.—GENERAL SCHOLIUM.

IN what I have done hitherto in this book by way of illustration of the general theorem of functional projections, I have imagined to be possible that which is really not so, and I have used fractional powers of -1, as if they had real existence in arithmetic. If the reader of this book has perused the first book, I flatter myself he will not be shocked by this mode of analysis; but, as it is my wish to keep this book distinct from that, I shall endeavour to explain the real arithmetical propriety, and object, of what is here delivered, from the principles of functional projections themselves; and, for this purpose, I shall offer the observation, that if radius be represented by unity, that $\overline{\cos.\text{ of }\overline{m}+\sqrt{\varrho-1}.\text{sine of }m+A.\varrho}\times\overline{\cos.\ n+\sqrt{\varrho-1}.\text{sine of }n}=\cos.\text{ of }\overline{m+n}+\sqrt{\varrho-1}.\text{sine of }\overline{m+n}+B.\varrho$; in which B will not contain any negative power of ϱ, if A does not; that is to say, provided we do not develope the expressions by means of series. It is also proper to add, that I use the term direct representation in a line of a point out of it, for the projection of the point in the line, according to the character of projection used. By the term *indirect* representation, I mean the projection of this projection into another line; and this I call the first indirect representation or projection: and, if this be again projected into another line, I call the third projection the second indirect projection. The advantage we have derived from the imaginary projection arises from the fact, that, if the thing were possible, the direct and indirect representation in a line of a point out of it would coincide.

No. 2.—I am now prepared to consider a functional representation which is not imaginary, whose character is, that, if G″ [*Fig.* 23] be the direct representation in the line PAP′ of the point G out of it, and GM be perpendicular to PAP′, cutting it in M, we have $MG'=\sqrt{\varrho-1}.MG$; and if AG be joined, we shall have $AG'=AG$ (cos. of GAM$+\sqrt{\varrho-1}$. sine of GAM). Now, suppose in the right lines Ag, Ag′, Ag″,

&c. g be the direct representation of G, g' the direct representation of g, that is the first indirect representation of G; g'' the direct representation of g', that is the second indirect representation of G, &c., all of the same character as that just mentioned, we shall have, by the general theorem, $Ag = AG \times (\text{cos. of } \overline{GAg} + \sqrt{\varrho - 1}.\text{sine of } GAg)$, $Ag' = Ag \times (\text{cos. of } gAg' + \sqrt{\varrho - 1}.\text{sine of } gAg')$, &c.; hence, by continuing the series of these equations, and keeping in view the observation offered in this article, we see that, if G′ in the right line PAP′ be one of the series of indirect representation of G, we shall have $AG' = AG.(\text{cos. of } GAG' + \sqrt{\varrho - 1}.\text{sine of } GAG' + B.\varrho)$; in which B is a finite function which does not contain (without developement in series) a negative power of ϱ; but, G″ being the direct representation of the same character, we have, as already stated, $AG' = AG.(\text{cos. of } GAG' + \sqrt{\varrho - 1}.\text{sine of } GAG')$; therefore G″G′, which is the distance between the indirect representations in AP′ of G, and the direct representation thereof, is $= AG.B.\varrho$ (B being independent of AG) that is, an expression of which every term is concerned with ϱ.

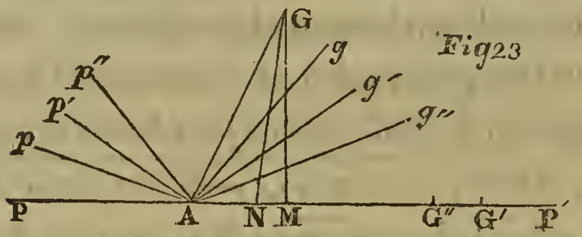

Fig. 23

No. 3.—Keeping to the same character of representation, suppose [*Fig.* 23] the angles P*Ap*, *pAp′*, *p′Ap″*, &c., all equal to each other and equal to q, and that two right angles be denoted by unity, and P, *p′*, *p″*, &c. up to P′ be the representations of P, *p*, *p′*, &c. in A*p*, A*p′*, A*p″*, &c.; put also $\mu = \text{cos. of } q + \sqrt{\varrho - 1}\,.\,\text{sine of } q$, consequently $Ap = AP.\mu$, $Ap' = Ap.\mu = AP.\mu^2$, $Ap'' = AP.\mu^3$, &c.; and, if $\frac{1}{q}$ be a whole number, which I shall suppose it to be, we have $AP' = AP \times \mu^{\frac{1}{q}}$. But, according to the character of representation here used, P is the direct representation of itself in the line PP′; and therefore, from No. 2 of this article, $PP' = AP.B.\varrho$; an expression in which every term is concerned with ϱ, B being independent of AP; therefore $AP.B.\varrho = AP \times (1 + \mu^{\frac{1}{q}})$; and therefore $\mu^{\frac{1}{q}} = B.\varrho - 1$.

No. 4.—Again, suppose the angle GAG′ divided into a certain whole number of equal parts GAg, gAg', &c. $=q$, that the angle GAG′$=p$, and that g, g', g'', &c., G′ are respectively the direct representations of G, g, g' &c. in Ag, Ag', Ag'', &c.; also that G″ is the direct representation of G in AP′; and we have, by No. 2, G″G′=AG.R. If R be put for a certain function (discoverable if necessary from what is laid down in No. 2) of ϱ, of which every term is independent of AG, but concerned with ϱ, we also have AG′=AG.$\mu^{\frac{p}{q}}$ and AG′=AM+$\sqrt{\varrho-1}$.GM; consequently AG.$\mu^{\frac{p}{q}}$, or its equal, from substituting for the value of $\mu^{\frac{1}{q}}$ found above, AG$.\overline{B.\varrho-1}^{p}$ =AM+GM $\sqrt{\varrho-1}$+AG.R. If ϱ be taken equal to o, R will $=o$, and we shall have the imaginary theorem AG$.\overline{-1}^{p}$=AM+GM$\sqrt{-1}$, as given by the imaginary projection; but, whether we use the one or the other of these two theorems, the operation must be equally legitimate in any research; as ϱ, being arbitrary, must vanish after the operation.

In AP′ between A and M take a point N, and join GN, put angle GNM$=n$, and we have, in a similar manner to what is above shewn, GN$.\overline{B.\varrho-1}^{n}$ =NM+GM$.\sqrt{\varrho-1}$+ a function of which every term is multiplied by ϱ, n being supposed a multiple of q; hence, from the two, we get AG$.\overline{B.\rho-1}^{p}$ =AN+NG$.\overline{B.\varrho-1}^{n}$ + a function of which every term is multiplied by ϱ; and thus we may go on to other cases.

No. 5.—If q be supposed infinitely small, and $c = 3.1416$, we have $\mu=1+\sqrt{\varrho-1}\cdot qc$ and $\mu^{\frac{1}{q}} = \overline{1+\sqrt{\varrho-1}\cdot qc}^{\frac{1}{q}} = \overline{1+q}^{\frac{c}{q}\sqrt{\varrho-1}}$, in virtue of q being infinitely small; but $\overline{1+q}^{\frac{1}{q}}$ will then be the number whose hyperbolical logarithm is 1; put it $=\varepsilon$, and we have $\mu^{\frac{1}{q}} = \varepsilon^{c\sqrt{\varrho-1}}$; and consequently, in virtue of the equation $AG.\mu^{\frac{p}{q}} = AG$ (cos. of $GAM + \sqrt{\varrho-1}$. sine of $GAM+R$), immediately deducible from one given above, we have, if z be put equal to the arc corresponding to angle GAG' to radius unity, $\varepsilon^{z\sqrt{\varrho-1}} =$ cos. of $z + \sqrt{\varrho-1}$. sine of $z +$ a function of which every term is multiplied by ϱ.

Note.—It is proper to caution some of my Readers, that though $\overline{-1}|^{n}$ is equal to $\overline{-1}|^{p}$ raised to the $\frac{n}{p}$ power; still, for instance, if p were taken equal to 2, it would not be proper to take $\overline{1}|^{\frac{1}{2}n}$ for $\overline{-1}|^{n}$, as the two are not identical, inasmuch as the former contains roots which the latter does not: and the same observation is applicable to other values of p. And, moreover, it is to be noticed that, although $\overline{-1}|^{n}$ and $\overline{-1}|^{-n}$ contain exactly the same radicals, still, in a formula containing the two expressions, it would be improper to use the one for the other; for, from what I have shewn above, we know how to find the roots of $\overline{-1}|^{n}$ and of $\overline{-1}|^{-n}$ by considering unity to represent $2 + 8\pi$ right angles, π being any whole number at pleasure; and we see that for one and the same whole number, taken for π, we have different values for $\overline{-1}|^{n}$ and $\overline{-1}|^{-n}$.

END OF TRACT II. BOOK II. OF IMAGINARY QUANTITIES.

ERRATA.

Page 10, line 6 from bottom, for DP read BP. Page 12, line 5 from bottom, for $a.PQ=)$ read $a.PQ)=$. Page 14, line 8, insert a comma before the second AG. Page 19, line 2 from bottom, insert *and* before sine. Page 20, line 6, insert *may* after PB.

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